SINGULAR LINE COMPENSATION IN THE PARAMETER PLANE

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THESIS

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ABSTRACT

Previous work on parameter plane, three dimensional parameter space, and singular lines in the parameter plane are reviewed.

A general concept of n-dimensional parameter space is hypothesized whereby the parameter plane becomes a special case of the general hypothesis. By the same argument the singular line is shown to be a special case of the singular hyperplane.

Existence criteria for singular lines are established, and compensation methods for creating singular lines in non-singular systems are derived and used.

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CHAPTER I

INTRODUCTION

The analysis of linear control systems by use of the Laplace Transforms of the systems' characteristic differential equations is adequately accomplished by root locus or frequency response methods when all coefficients of the complex variable s are real constants.

These same methods are also sufficient to effect analysis when the coefficients are functions of a single real variable K. The general characteristic equation

$$f(s) = \sum_{k=0}^{n} a_k s^k = 0 \qquad (1\text{-}1)$$

may be expressed as

$$f(s) = \sum_{k=0}^{n} [b_k(K)+c_k]s^k = 0 \ , \qquad (1\text{-}2)$$

from which follows

$$\frac{\sum_{k=0}^{n} b_k(K)s^k}{\sum_{k=0}^{n} c_k s^k} = -1 \ , \qquad (1\text{-}3)$$

which is then in the proper form for either root locus or frequency response analysis.

However, many modern control systems, although still linearly conceived or approximated, cannot be handled by the above somewhat "classical" techniques. The complexities accompanying the advancing "state of the art" in control system and compensation design often result in characteristic equation coefficients that are functions of two or more variables.

In 1959, Mitrovic introduced a method [1] for determining the roots of the characteristic equation when the coefficients of the two lowest ordered terms vary independently.

A point in the s-plane,

$$s = -\sigma \mp j\omega \tag{1-4}$$

may be specified in polar form as

$$s = -\zeta\omega_n \mp j\omega_n\sqrt{1-\zeta^2}, \tag{1-5}$$

where ω_n is the radial distance from the origin to the point and ζ is the cosine of θ, the angle between the real axis and ω_n.

Since

$$\zeta\omega_n = \cos\theta \tag{1-6}$$

and

$$\omega_n\sqrt{1-\zeta^2} = \sin\theta, \tag{1-7}$$

equation 1-5 may be expressed as

$$s = \omega_n(\cos\theta + j\sin\theta); \tag{1-8}$$

and

$$s^k = \omega_n^k(\cos k\theta + j\sin k\theta). \tag{1-9}$$

As previously defined, for left-hand plane roots

$$\theta = \cos^{-1}(-\zeta); \tag{1-10}$$

therefore

$$\cos k\theta = \cos k[\cos^{-1}(-\zeta)], \tag{1-11}$$

which can be expressed as a Chebyshev function of the first kind, $T_k(-\zeta)$, and

$$\sin k\theta = \sin k[\cos^{-1}(-\zeta)], \tag{1-12}$$

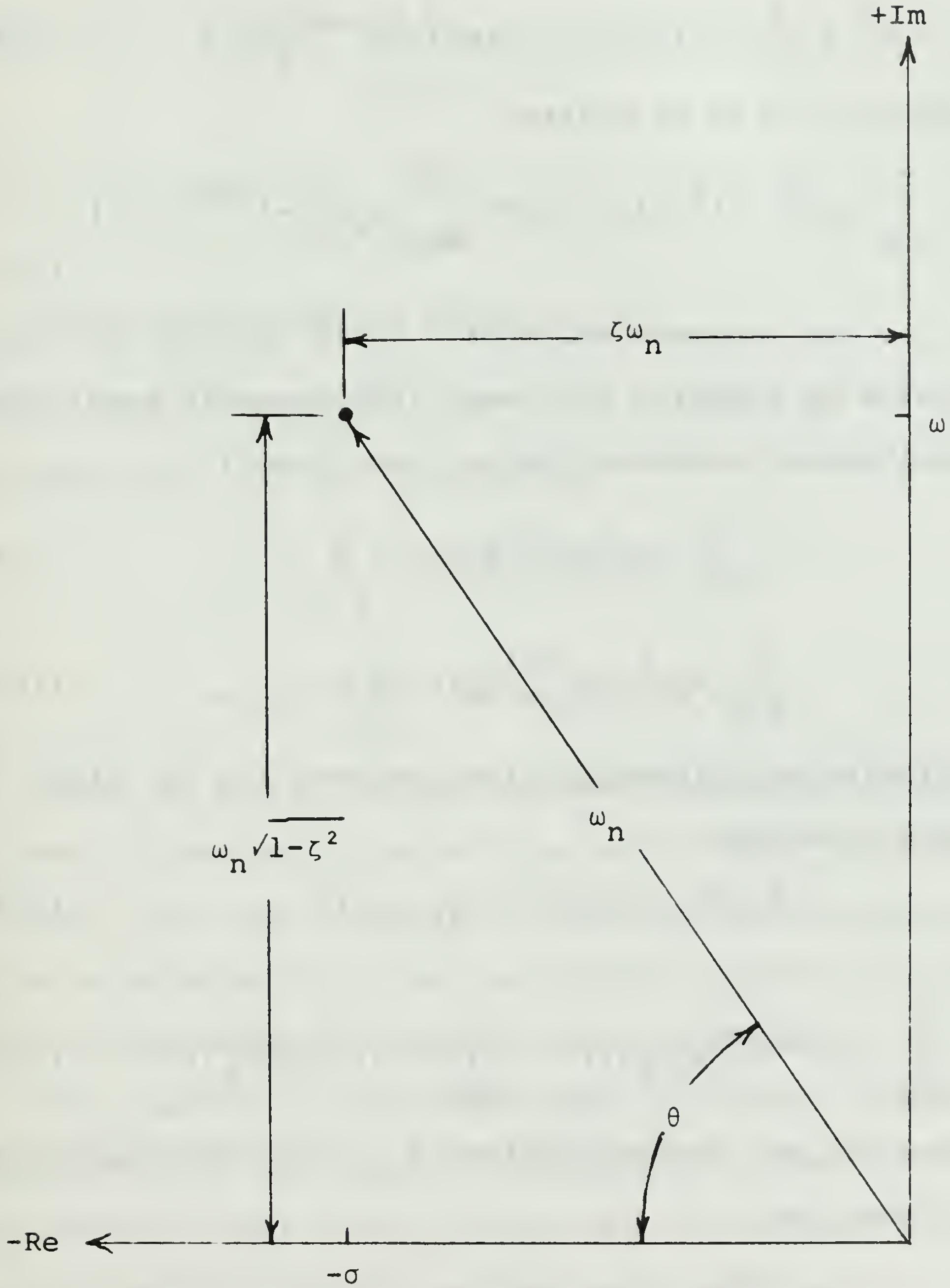


Fig. 1.1 A Point in the s-Plane

which can be expressed by a Chebyshev function of the second kind, $\sin\theta U_k(-\zeta)$. This permits equation 1-9 to be written

$$s^k = \omega_n^k[T_k(-\zeta)+j\sin\theta U_k(-\zeta)] \tag{1-13}$$

$$s^k = \omega_n^k[(-1)^k T_k(\zeta)+j\sin\theta(-1)^{k+1}U_k(\zeta)] \tag{1-14}$$

and equation 1-1 to be written

$$f(s) = \sum_{k=0}^{n} a_k\omega_n^k(-1)^k T_k(\zeta)+j\sin\theta \sum_{k=0}^{n} a_k\omega_n^k(-1)^{k+1}U_k(\zeta) = 0. \tag{1-15}$$

Since, for any non-zero value of θ, both the real and imaginary parts of equation 1-15 must independently equal zero, two simultaneous equations for a_k are formed:

$$\sum_{k=0}^{n} a_k\omega_n^k(-1)^k T_k(\zeta) = 0 \tag{1-16}$$

$$\sum_{k=0}^{n} a_k\omega_n^k(-1)^{k+1}U_k(\zeta) = 0 . \tag{1-17}$$

The following relationship exists between the two kinds of Chebyshev functions:

$$T_k(\zeta) = \zeta U_k(\zeta) - U_{k-1}(\zeta); \tag{1-18}$$

also

$$\sum_{k=0}^{n} (-1)^{k+1}a_k\omega_n^k U_k(\zeta) = -\sum_{k=0}^{n} (-1)^k a_k\omega_n^k U_k(\zeta); \tag{1-19}$$

therefore for any non-zero value of ζ, the simultaneous equations reduce to

$$\sum_{k=0}^{n} (-1)^k a_k\omega_n^k U_{k-1}(\zeta) = 0 \tag{1-20}$$

$$\sum_{k=0}^{n} (-1)^k a_k\omega_n^k U_k(\zeta) = 0 . \tag{1-21}$$

$U_k(\zeta)$ can be obtained for all values of k from the recursion formula

$$U_{k+1}(\zeta) = 2\zeta U_k(\zeta) - U_{k-1}(\zeta), \tag{1-22}$$

where, for all values of ζ,

$$U_{-1}(\zeta) = -1 \tag{1-23}$$

$$U_0(\zeta) = 0 \tag{1-24}$$

$$U_1(\zeta) = 1 . \tag{1-25}$$

Equation 1-1 can be rewritten as

$$f(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots a_2 s^2 + A_1 s + A_0 = 0, \tag{1-26}$$

and equations 1-20 and 1-21 become

$$-A_0 + \sum_{k=2}^{n} (-1)^k a_k \omega_n^k U_{k-1}(\zeta) = 0 \tag{1-27}$$

$$-\omega_n A_1 + \sum_{k=2}^{n} (-1)^k a_k \omega_n^k U_k(\zeta) = 0 . \tag{1-28}$$

Thus, A_0 and A_1 become the coordinate axes of the Mitrovic Plane; and lines of constant ζ and ω_n, along with constant real root lines from the direct substitution of $(-\sigma)$ for s in equation 1-26, can be plotted, giving the solution of the characteristic equation for any point (A_0, A_1).

This method was later extended by Elliott, Thaler, and Heseltine [2,3] for specific coefficient pairs, and then generalized by Siljak [2,4] for any pair of variable coefficients in the Coefficient Plane method, where equation 1-1, when written as

$$f(s) = A_l s^l + A_m s^m + \sum_{\substack{k=0 \\ k \neq l,m}}^{n} a_k s^k = 0, \tag{1-29}$$

results in the set of simultaneous equations in (A_1, A_m)

$$(-1)^1 A_1 \omega_n^1 U_{1-1}(\zeta) + (-1)^m A_m \omega_n^m U_{m-1}(\zeta) + \sum_{\substack{k=0 \\ k \neq 1,m}}^{n} (-1)^k a_k \omega_n^k U_{k-1}(\zeta) = 0 \tag{1-30}$$

$$(-1)^1 A_1 \omega_n^1 U_1(\zeta) + (-1)^m A_m \omega_n^m U_m(\zeta) + \sum_{\substack{k=0 \\ k \neq 1,m}}^{n} (-1)^k a_k \omega_n^k U_k(\zeta) = 0. \tag{1-31}$$

Although this is an improvement of Mitrovic's original method, it is still limited by the requirement that only two coefficients of the characteristic equation contain variable elements. However, in 1964 Siljak [5] extended this process into the Parameter Plane, where two variable parameters can appear in any of the coefficients of the characteristic equation. In the linear case,

$$a_k = b_k\alpha + c_k\beta + f_k, \tag{1-32}$$

where α and β are the adjustable parameters. Hence, equation 1-1 becomes

$$f(s) = \alpha \sum_{k=0}^{n} b_k s^k + \beta \sum_{k=0}^{n} c_k s^k + \sum_{k=0}^{n} f_k s^k = 0; \tag{1-33}$$

and equations 1-20 and 1-21 become

$$\alpha \sum_{k=0}^{n} (-1)^k b_k \omega_n^k U_{k-1}(\zeta) + \beta \sum_{k=0}^{n} (-1)^k c_k \omega_n^k U_{k-1}(\zeta) + \sum_{k=0}^{n} (-1)^k f_k \omega_n^k U_{k-1}(\zeta) = 0 \tag{1-34}$$

$$\alpha \sum_{k=0}^{n} (-1)^k b_k \omega_n^k U_k(\zeta) + \beta \sum_{k=0}^{n} (-1)^k c_k \omega_n^k U_k(\zeta) + \sum_{k=0}^{n} (-1)^k f_k \omega_n^k U_k(\zeta) = 0 . \tag{1-35}$$

For notational ease,

$$X_1 = \sum_{k=0}^{n} (-1)^k x_k \omega_n^k U_{k-1}(\zeta) \qquad (1-36)$$

$$X_2 = \sum_{k=0}^{n} (-1)^k x_k \omega_n^k U_k(\zeta) \ , \qquad (1-37)$$

and

$$B_1\alpha + C_1\beta + F_1 = 0 \qquad (1-38)$$

$$B_2\alpha + C_2\beta + F_2 = 0, \qquad (1-39)$$

which are solved by standard methods, giving constant ζ, ω_n, and σ lines in the α-β Parameter Plane for the solution of the characteristic equation.

In 1968 Cadena [6] extended the parameter plane concept to three independent parameters, where

$$a_k = b_k\alpha + c_k\beta + d_k\gamma + f_k. \qquad (1-40)$$

The parametric simultaneous equations then become

$$B_1\alpha + C_1\beta + D_1\gamma + F_1 = 0 \qquad (1-41)$$

$$B_2\alpha + C_2\beta + D_2\gamma + F_2 = 0, \qquad (1-42)$$

which when solved give the line of intersection of two planes in 3-parameter space. By an ingenious device of rotating the γ coordinate by 90° to form the $(-\beta)$ ordinate in the α-β plane, Cadena, by graphical projections, reduces the constant ζ, ω_n, and σ planes to families of lines in the α-β plane -- each line representing a specific value of γ. In the same work, it is shown that for an n-parameter system, n-roots may be fixed by the adjustment of each parameter to a specific value. In this case the remaining roots are then determined by factoring the quotient which results from the division of the characteristic equation by the product of the specified roots.

What emerges from this "reduced characteristic equation" concept is that a designer has at his disposal a "degree of freedom" for each variable parameter in his system. If, for example, the system contains three parameters, three roots may be fixed by rigid adjustment of all three parameters; if only two roots are specified (e.g., a complex conjugate pair), the reduced characteristic equation then becomes a function of one parameter (the remaining degree of freedom), and ordinary root locus methods determine the remaining roots; if only one real root is specified, the reduced characteristic equation is then a two parameter case, leading to root solution by Parameter Plane methods.

Thus Parameter Plane techniques have become a powerful tool for both the analysis and synthesis of modern complex control systems.

CHAPTER II

THE SINGULAR LINE

In the preceding chapter it was shown that constant ζ, ω_n, and σ lines could be mapped in the α-β plane by the simultaneous solutions of equations 1-38 and 1-39. Using Cramer's Rule, points $M^1(\zeta,\omega_n)$ can be mapped into a point $M(\alpha,\beta)$ by

$$\alpha = \frac{\begin{vmatrix} -F_1 & C_1 \\ -F_2 & C_2 \end{vmatrix}}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \frac{C_1F_2 - C_2F_1}{B_1C_2 - B_2C_1} \tag{2-1}$$

and

$$\beta = \frac{\begin{vmatrix} B_1 & -F_1 \\ B_2 & -F_2 \end{vmatrix}}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \frac{B_2F_1 - B_1F_2}{B_1C_2 - B_2C_1}\,, \tag{2-2}$$

the point, M, being the intersection of constant ζ lines and constant ω_n lines. By this method the roots of characteristic equations can be determined from the output graphs of various Parameter Plane computer programs, using linear interpolation, if necessary, to find all roots at any point $M(\alpha,\beta)$.

An obvious limitation of this method is that the determinant of the coefficient matrix be non-singular, or non-zero.

Although this limitation existed, it was dismissed as being of little significance until 1967, when Bowie [7], in attempting to solve a sixth order characteristic equation, found, that by selecting various points $M(\alpha,\beta)$, only certain roots could be found using Parameter Plane methods. However, by substituting the co-ordinates of the point $M(\alpha,\beta)$ into the characteristic equation and factoring, the undetermined roots were found to be complex conjugate pairs. Moreover, many points $M(\alpha,\beta)$ were found to have the same complex pair in common; these common root pair points formed a straight line in the α-β plane which was the locus of a constant ζ-constant ω_n pair.

Upon further investigation of this phenomenon, Bowie discovered that lines of constant values of ζ, ω_n, which he called singular lines, existed whenever equations 1-38 and 1-39 were linearly dependent.

If the coefficient matrix of equations 1-38 and 1-39 is expanded in terms of ζ and ω_n, an infinite number of singular lines can be found by setting the determinant to zero and solving for either ζ or ω_n in terms of the other. For an equation of order n, there are a maximum of 2(n-1) values of ω_n which produce singular lines for each of the infinite number of values of ζ; conversely, (n-1) values of ζ will produce singularities for each value of ω_n.

Bowie illustrated this principle with two examples, which will be reproduced at this point:

<u>Example I</u>

Consider the characteristic equation

$$f(s) = 0.04s^4 + 0.34s^3 + (0.2\alpha+1.12)s^2 + (0.5\alpha+\beta+1.7)s + 2\beta + 1 = 0. \quad (2\text{-}3)$$

For ease of coefficient identification, display the terms as

k	4	3	2	1	0
b_k	-	-	0.2	0.5	-
c_k	-	-	-	1	2
f_k	0.04	0.34	1.12	1.7	1

Then, if $\zeta = 0.5$,

$$B_1 = 0.2\omega_n^2 \qquad C_1 = -2$$
$$B_2 = -0.5\omega_n + 0.2\omega_n^2 \qquad C_2 = -\omega_n \; ; \tag{2-4}$$

and

$$B_1C_2 - B_2C_1 = -0.2\omega_n^3 + 0.4\omega_n^2 - \omega_n = 0. \tag{2-5}$$

Reducing,

$$\omega_n^2 - 2\omega_n + 5 = 0; \tag{2-6}$$

and

$$\omega_n = 1 \pm j2 \; . \tag{2-7}$$

Since no real values of ω_n exist, no singular line will exist for this characteristic equation for the value $\zeta = 0.5$.

<u>Example II</u>

$$f(s) = s^6 + 80s^5 + (20\alpha+1600)s^4 + 840\alpha s^3 + (1600\alpha+400\beta)s^2$$
$$+ 1600\beta s + 1600\beta = 0. \tag{2-8}$$

Displaying terms,

k	6	5	4	3	2	1	0
b_k	-	-	20	840	1600	-	-
c_k	-	-	-	-	400	1600	1600
f_k	1	80	1600	-	-	-	-

For $\zeta = 0.5$,

$$B_1 = 1600\omega_n^2 - 840\omega_n^3 \qquad C_1 = -1600 + 400\omega_n^2$$
$$B_2 = 1600\omega_n^2 - 20\omega_n^4 \qquad C_2 = -1600\omega_n + 400\omega_n^2 \tag{2-9}$$

and

$$\omega_n^4 - 42\omega_n^3 + 164\omega_n^2 - 320\omega_n + 320 = 0. \tag{2-10}$$

Solving,

$$\omega_n = 2.1115,\ 37.889,\ \text{and a complex pair.} \tag{2-11}$$

Therefore, for $\zeta = 0.5$, there exist two values of ω_n — 2.1115 and 37.889, which will produce singular lines for equation 2-8.

Once the conditions for singularity have been met, the singular line may be represented by the equation

$$\beta = -\frac{B_1}{C_1}\alpha - \frac{F_1}{C_1} \equiv -\frac{B_2}{C_2}\alpha - \frac{F_2}{C_2}\ . \tag{2-12}$$

Cadena [6] investigated the singular line when the characteristic equation coefficients are functions of three variables.

Since the solution of equations 1-41 and 1-42 is a line in three dimensional space for a ζ-ω_n pair, the definition singular line may be loosely applied. Where that line intersects the α-β-co-ordinate plane is the normal two dimensional solution (or singular point) of equations 1-38 and 1-39, since $\gamma = 0$. This actually reduces to his "degree of freedom" concept, whereby two roots may be specified by specifying the proper values of two of the variable parameters.

A special case of the three dimensional singular line is when it happens to lie in one of the co-ordinate planes or in a plane parallel to one of the co-ordinate planes. When this occurs, the singular line of equation 2-12 exists for the specific value of γ.

An illustrative example of the foregoing was given by Cadena:

<u>Example III</u>

Given

$$f(s) = s^4 + (10\alpha+10)s^3 + (40\alpha+5\beta+10\gamma+30)s^2 + (80\alpha+10\beta+252)s + 70\alpha + 10\beta + 25\gamma + 340 = 0, \quad (2\text{-}13)$$

it is desired that a pair of complex conjugate roots be located at $\zeta = 0.5$, $\omega_n = 2$. Division of equation 2-13 by the product of the desired roots, $s^2 + 2s + 4$, yields a reduced characteristic equation in α, β, γ and a remainder which must equal zero for the division to be exact.

$$(-20\gamma+200)s + (-10\alpha-10\beta-15\gamma+300) = 0 \quad (2\text{-}14)$$

or

$$R_1 s + R_0 = 0 \ . \quad (2\text{-}15)$$

Obviously, R_1, and R_0 must simultaneously be zero, so

$$\gamma - 10 = 0 \quad (2\text{-}16)$$

and

$$10\alpha + 10\beta + 15\gamma - 300 = 0. \quad (2\text{-}17)$$

This yields the singular line

$$\alpha + \beta - 15 = 0 \tag{2-18}$$

in the plane $\gamma = 10$.

Perhaps a more illuminating manner in which to view this problem is to display the coefficients in tabular form as was done for the two-dimensional examples. Thus

k	4	3	2	1	0
b_k	-	10	40	80	70
c_k	-	-	5	10	10
d_k	-	-	10	-	25
f_k	1	10	30	252	340

Then, for $\zeta = 0.5$, $\omega_n = 2$,

$$\begin{array}{llll} B_1 = 10 & C_1 = 10 & D_1 = 15 & F_1 = -300 \\ B_2 = 0 & C_2 = 0 & D_2 = 40 & F_2 = -400 \end{array} \tag{2-19}$$

and

$$10\alpha + 10\beta + 15\gamma - 300 = 0 \tag{2-20}$$

$$40\gamma - 400 = 0. \tag{2-21}$$

This simultaneous system is the singular line formed by the intersection of the two planes described by equations 2-20 and 2-21. The singular line is

$$\alpha + \beta - 15 = 0 \tag{2-22}$$

in the plane

$$\gamma = 10. \tag{2-23}$$

Now, if $d_1 = 10$,

$$D_2 = 20; \tag{2-24}$$

and

$$10\alpha + 10\beta + 15\gamma - 300 = 0 \tag{2-25}$$

$$20\gamma - 400 = 0. \tag{2-26}$$

The singular line is then

$$\alpha + \beta = 0 \tag{2-27}$$

in the plane

$$\gamma = 20. \tag{2-28}$$

If now $d_1 = d_3 = d_4 = 10$,

$$D_1 = 95 \quad D_2 = 180; \tag{2-29}$$

$$10\alpha + 10\beta + 95\gamma - 300 = 0 \tag{2-30}$$

$$180\gamma - 400 = 0; \tag{2-31}$$

and the singular line becomes

$$\alpha + \beta - \frac{80}{9} = 0 \tag{2-32}$$

in the plane

$$\gamma = \frac{20}{9}. \tag{2-33}$$

By now it is apparent that any variation of coefficients d_k will result in different singular lines in different planes parallel to the α-β plane.

What then happens if coefficients b_k and/or c_k are varied? Let $c_3 = c_4 = 10$. Then

$$C_1 = -70 \quad C_2 = 160 \tag{2-34}$$

and

$$10\alpha - 70\beta + 15\gamma - 300 = 0 \tag{2-35}$$

$$-160\beta + 40\gamma - 400 = 0. \tag{2-36}$$

The singular line for $\zeta = 0.5$, $\omega_n = 2$ now exists in α, β, γ-space at the intersection of the two planes described by equations 2-35 and 2-36. Of what use is this information? If equation 2-36 is solved for γ with the result then substituted into equation 2-35, the singular line is then "mapped" into the α-β plane as

$$\alpha - \beta - 15 = 0 \qquad (2\text{-}37)$$

via the "transform"

$$\gamma = 4\beta + 10. \qquad (2\text{-}38)$$

What does this mean? For any consistent three dimensional set of equations 1-41 and 1-42, a two dimensional singular line can be realized by placing an appropriate constraint upon the third parameter.

For further examination, let $c_2 = 0$. Then

$$C_1 = -10 \qquad C_2 = -20; \qquad (2\text{-}39)$$

$$10\alpha - 10\beta + 15\gamma - 300 = 0 \qquad (2\text{-}40)$$

$$-20\beta + 40\gamma - 400 = 0; \qquad (2\text{-}41)$$

and

$$4\alpha - \beta - 60 = 0 \qquad (2\text{-}42)$$

if

$$\gamma = \tfrac{1}{2}\beta + 10. \qquad (2\text{-}43)$$

Let $b_4 = 10$. Then

$$B_2 = 160; \qquad (2\text{-}44)$$

$$10\alpha + 10\beta + 15\gamma - 300 = 0 \qquad (2\text{-}45)$$

$$160\alpha + 40\gamma - 400 = 0; \qquad (2\text{-}46)$$

and

$$50\alpha - 10\beta + 150 = 0 \qquad (2\text{-}47)$$

if

$$\gamma = -4\alpha + 10. \tag{2-48}$$

What has been demonstrated in the preceding examples is the existence of a powerful tool for the synthesis of self-adaptive systems or, for that matter, any system for which a characteristic response is desired. How this tool is used will be the subject of further discussion later in this thesis.

CHAPTER III

N-DIMENSIONAL PARAMETER SPACE

Consider a set of n consistent, independent simultaneous equations where $n = N$:

$$
\begin{aligned}
&B_1\alpha + C_1\beta + D_1\gamma + E_1\delta + \ldots + N_1\nu + P_1 = 0 \\
&B_2\alpha + C_2\beta + D_2\gamma + E_2\delta + \ldots + N_2\nu + P_2 = 0 \\
&\vdots \\
&B_n\alpha + C_n\beta + D_n\gamma + E_n\delta + \ldots + N_n\nu + P_n = 0.
\end{aligned}
\tag{3-1}
$$

The solution of the system of equations 3-1 is the set of values obtained for the variables α, β, γ, δ, ..., ν, which defines a point, which is a zero (n-n) dimensional entity, in n-dimensional space.

If now only (n-1) independent equations in $n = N$ variables can be obtained, the set is reduced to

$$
\begin{aligned}
&B_1\alpha + C_1\beta + D_1\gamma + E_1\delta + \ldots + N_1\nu + P_1 = 0 \\
&B_2\alpha + C_2\beta + D_2\gamma + E_2\delta + \ldots + N_2\nu + P_2 = 0 \\
&\vdots \\
&B_{n-1}\alpha + C_{n-1}\beta + D_{n-1}\gamma + E_{n-1}\delta + \ldots + N_{n-1}\nu + P_{n-1} = 0.
\end{aligned}
\tag{3-2}
$$

The solution to set 3-2 is a one [n-(n-1)] dimensional line defined by an arbitrary pair of the N variables.

Similarly, for (n-2) independent equations, the solution becomes a two [n-(n-2)] dimensional plane defined by an arbitrary trio of the N variables.

By induction it is clear that if only two independent equations describe a system of N variables, there results equations

$$B_1\alpha + C_1\beta + D_1\gamma + E_1\delta + \ldots + N_1\nu + P_1 = 0 \qquad (3\text{-}3)$$

$$B_2\alpha + C_2\beta + D_2\gamma + E_2\delta + \ldots + N_2\nu + P_2 = 0 \qquad (3\text{-}4)$$

whose solution is an (n-2) dimensional hyperplane defined by an arbitrary (n-1) set of the N variables.

If the coefficients B, C, D, E, ..., N, P are now functions of some constant pair (ζ,ω_n), the solution to equations 3-3 and 3-4 is an (n-2) dimensional hyperplane of constant ζ, ω_n. If, when all but two arbitrary variables are set to zero and equations 3-3 and 3-4 remain consistent and independent, the point of intersection of the (a-2) dimensional constant ζ, ω_n hyperplane with the plane of the two remaining co-ordinates is determined. If equations 3-3 and 3-4 become dependent, the intersection is a line; if consistent, no intersection exists.

For non-linear combinations of the N variables, the (n-2) dimensional hyperplane then becomes hyperbolic, parabolic, cubic, quartic, etc., in nature, which, in the independent and consistent case described in the preceding paragraph, gives rise to the possibility of multiple intersections with the two-dimensional co-ordinate planes.

Although extremely little of practical engineering value can be extracted from the preceding discussion, the concepts expressed are valid and are useful for an appreciation of the two parameter problem. Since mental imagery extends at best to a relatively poor perception of three dimensions, the root finding problem is limited at the present state of the art to the three dimensional case demonstrated by Cadena [6].

However, one limited use is the determination of necessary values of two parameters, say α and β, given arbitrary values of the remaining n parameters, for obtaining a desired ζ, ω_n pair.

Given a system described by the characteristic equation

$$f(s) = \sum_{k=0}^{n} a_k s^k = 0 \tag{3-5}$$

where

$$a_k = b_k\alpha + c_k\beta + d_k\gamma + e_k\delta + \ldots + p_k, \tag{3-6}$$

one arrives at equations 3-3 and 3-4, where all capital letter coefficients are functions of the desired ζ, ω_n pair. If variables γ, δ, ..., ν can be fixed or measured, the problem then becomes two dimensional in α and β, leading to solution by the methods outlined in chapter one. This is essentially what Cadena achieves by graphical means by folding the γ-coordinate into the α-β plane.

Figure 3.1 is illustrative of the three dimensional linear case. For the system

$$B_1\alpha + C_1\beta + D_1\gamma + F_1 = 0 \tag{3-7}$$

$$B_2\alpha + C_2\beta + D_2\gamma + F_2 = 0, \tag{3-8}$$

the constant ζ, ω_n [{n=3}-2] hyperplane is a line in three dimensional space.

For the case of non-linear combinations of variables, the ζ, ω_n lines become curves in space with the possibility

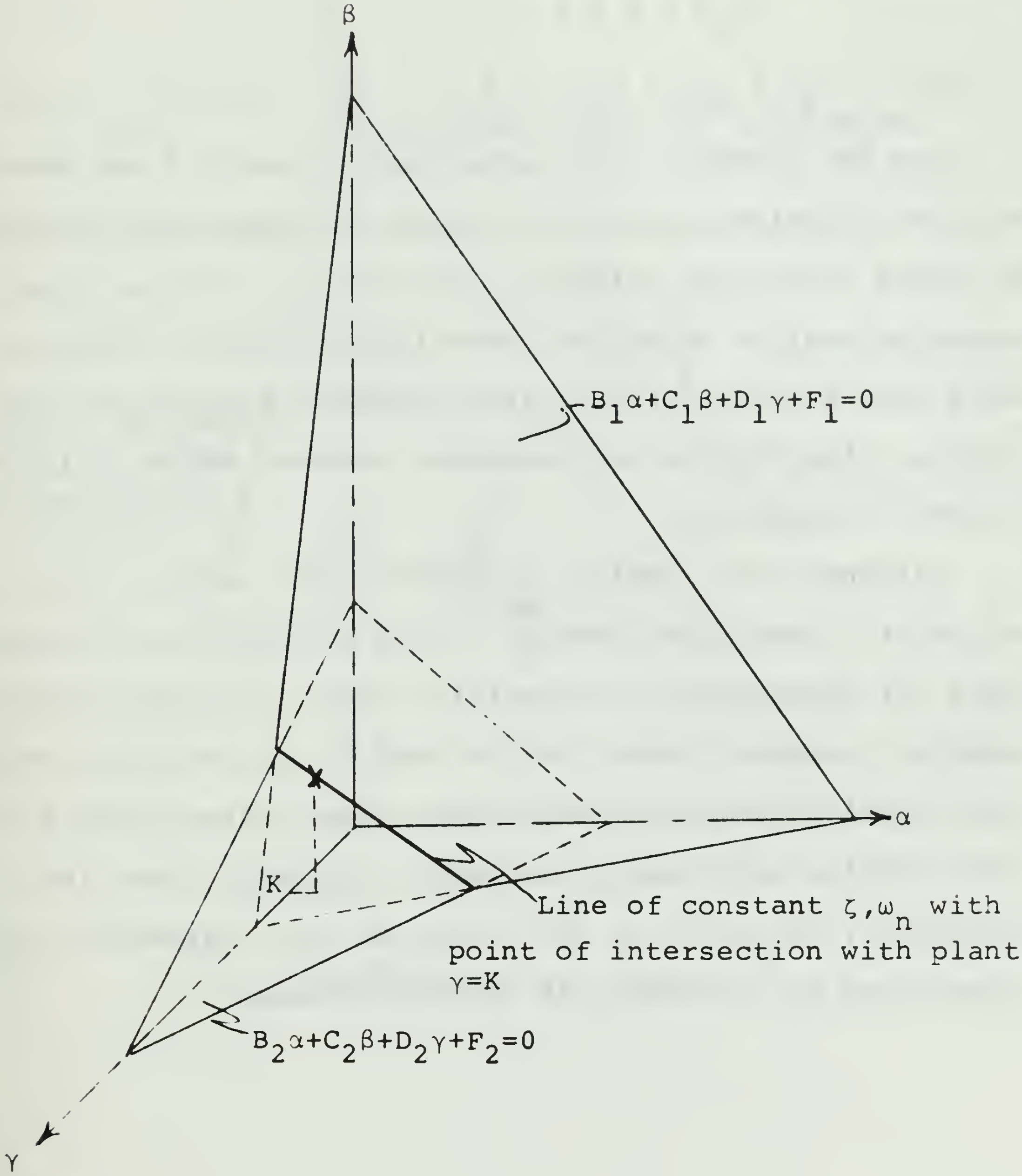


Fig. 3.1 Intersection of Two Planes in Three Parameter Space

of multiple intersections with constant γ-planes as illustrated in figure 3.2 for the system

$$B_1\alpha + C_1\beta + D_1\gamma + H_1 = 0 \qquad (3\text{-}9)$$

$$B_2\alpha + C_2\beta + D_2\gamma + E_2\alpha^2 + F_2\beta^2 + G_2\gamma^2 + H_2 = 0. \qquad (3\text{-}10)$$

For the condition when equations 3-3 and 3-4 are dependent, the solution is an (n-1) dimensional hyperplane described by either of the two "singular" equations. The same line of reasoning applies as in the "non-singular" case, resulting in the singular plane for the three parameter problem and the singular line for the two parameter problem, which is illustrated in figure 3.3.

Although the singular line is the most useful of all the "singular" hyperplanes because of the availability of graphical means for determining the remaining roots of a characteristic equation, singular theory can be readily applied to any multivariate self-adaptive system or any other system where a constant complex root pair is desired. For any system with n parameters, variation in any or all of (n-1) parameters may be compensated by adjustment of the nth parameter.

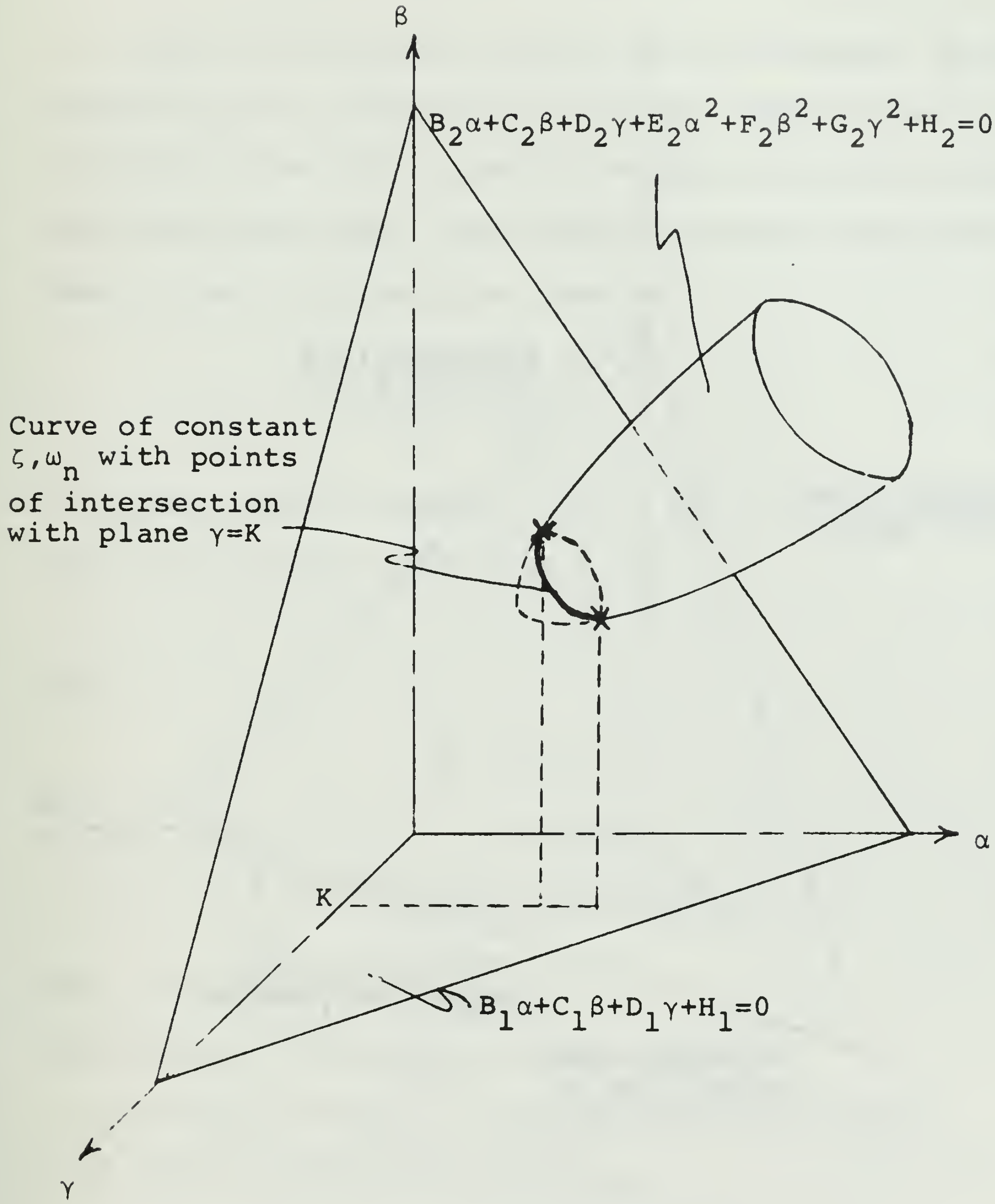


Fig. 3.2 Non-linear Intersection in Three Parameter Space

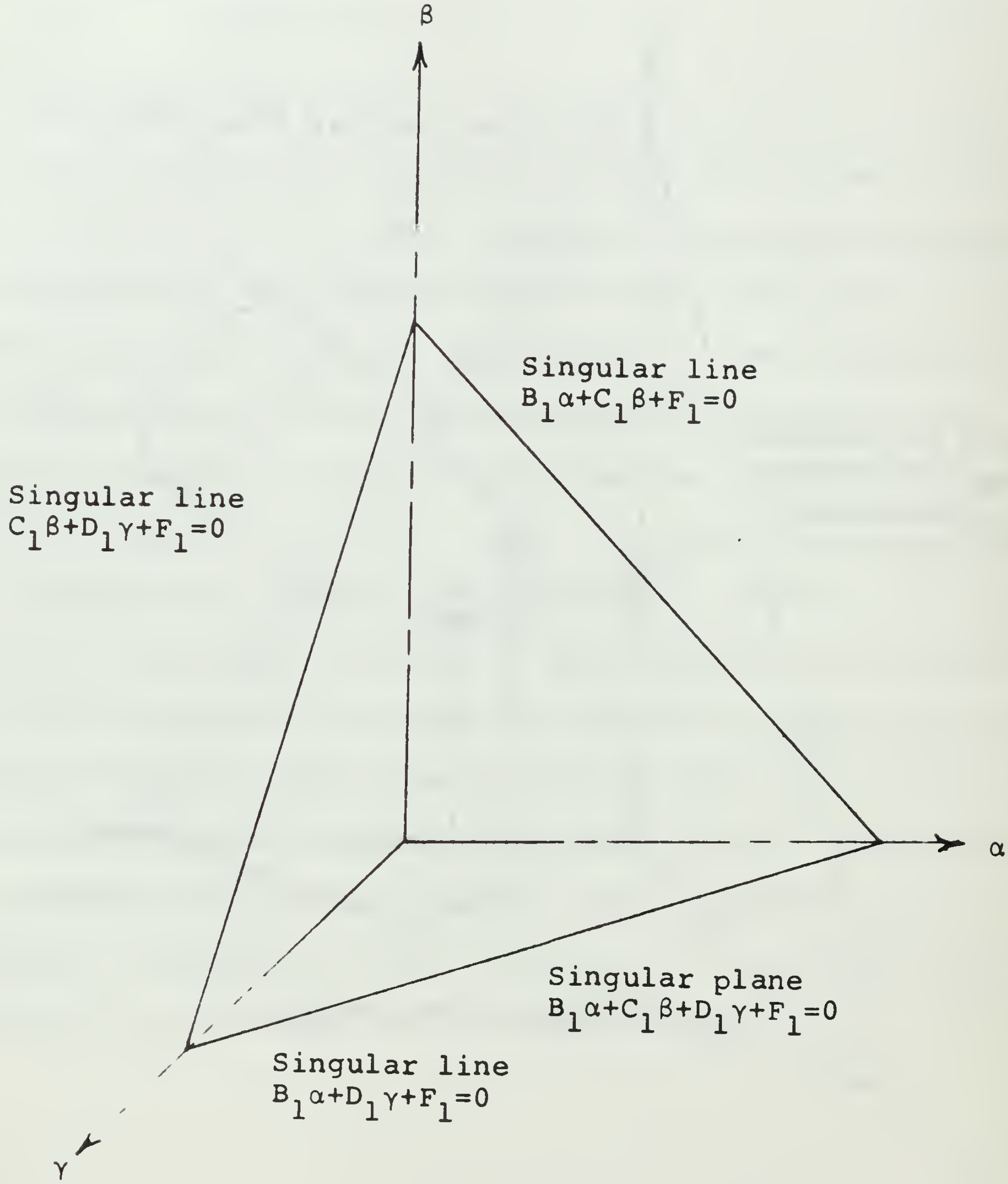


Fig. 3.3 Singular Plane

CHAPTER IV

INTRODUCTION TO COMPENSATION

Since the singular line can be an extremely useful system characteristic, the ability to design compensation to produce a singular line for a specific complex conjugate root pair becomes very desirable. The simplest mathematical scheme which comes to mind is that, for a system

$$B_1\alpha + C_1\beta + F_1 = 0 \qquad (4\text{-}1)$$

$$B_2\alpha + C_2\beta + F_2 = 0, \qquad (4\text{-}2)$$

a singular line is present if $B_1 = B_2$, $C_1 = C_2$, and $F_1 = F_2$. Using the general coefficient X, if

$$X_1 = X_2, \qquad (4\text{-}3)$$

then

$$X_1 - X_2 = 0, \qquad (4\text{-}4)$$

and

$$\sum_{k=0}^{n} (-1)^k x_k \omega_n^k [U_{k-1}(\zeta) - U_k(\zeta)] = 0, \qquad (4\text{-}5)$$

which, when expanded, becomes

$$\begin{aligned} &- x_0 + \omega_n x_1 + (1-2\zeta)\omega_n^2 x_2 + (-1-2\zeta+4\zeta^2)\omega_n^3 x_3 \\ &+ (-1+4\zeta+4\zeta^2-8\zeta^3)\omega_n^4 x_4 + (1+4\zeta-12\zeta^2-8\zeta^3+16\zeta^4)\omega_n^5 x_5 \\ &+ (1-6\zeta-12\zeta^2+32\zeta^3+16\zeta^4-32\zeta^5)\omega_n^6 x_6 \\ &+ (-1-6\zeta+24\zeta^2+32\zeta^3-80\zeta^4-32\zeta^5+64\zeta^6)\omega_n^7 x_7 \\ &+ (-1+8\zeta+24\zeta^2-80\zeta^3-80\zeta^4+192\zeta^5+64\zeta^6-128\zeta^7)\omega_n^8 x_8 \\ &+ (1+8\zeta-40\zeta^2-80\zeta^3+240\zeta^4+192\zeta^5-448\zeta^6-128\zeta^7+256\zeta^8)\omega_n^9 x_9 \\ &+ (1-10\zeta-40\zeta^2+160\zeta^3+240\zeta^4-672\zeta^5-448\zeta^6+1024\zeta^7+256\zeta^8 \\ &\quad -512\zeta^9)\omega_n^{10} x_{10} + \ldots = 0 \, . \end{aligned} \qquad (4\text{-}6)$$

If the coefficients of ζ^m for each term k are displayed as in Table 4-1, the result, neglecting the k = 0 row, which is (-1) for m = 0, is a lower left diagonal matrix with each coefficient repeating once in the same column. The repeating coefficient pairs alternate in sign as each column is formed. The values of the main diagonal-pair are $(-2)^m$, and the succeeding diagonal-pairs are formed by entering Table 4-2 with arguments l and m to obtain the necessary multipliers of the magnitude of the elements in the main diagonal-pair.

Table 4-2 is easily formed by first making elements l_0, m_j and l_i, m_0 equal to one; then

$$l_i,\ m_j = l_i,\ m_{j-1} + l_{i-1},\ m_j\ . \tag{4-7}$$

Therefore, by establishing the preceding conditions for all coefficients X of equations 4-1 and 4-2, compensation to produce a singular line can be effected.

Because plant gain is usually the available control to adjust, it is convenient to equate all terms x_0 in the form

$$x_0 = \omega_n x_1 + (1-2\zeta)\omega_n^2 x_2 + \ldots\ , \tag{4-8}$$

since gain for a type-one unity-feedback plant with no zeros, which can always be realized mathematically by block diagram algebra, is the sum of all the x_0 terms.

This compensation can be easily achieved with a cascaded component whose transfer function numerator contains the term $b_0\alpha + c_0\beta + f_0$ and whose denominator cancels the numerator of the uncompensated plant.

Coefficients of ζ^m

k \ m	0	1	2	3	4	5	6	7	8	9
0	-1									
1	1									
2	1	-2								
3	-1	-2	4							
4	-1	4	4	-8						
5	1	4	-12	-8	16					
6	1	-6	-12	32	16	-32				
7	-1	-6	24	32	-80	-32	64			
8	-1	8	24	-80	-80	192	64	-128		
9	1	8	-40	-80	240	192	-448	-128	256	
10	1	-10	-40	160	240	-672	-448	1024	256	-512
	l = 4		l = 3		l = 2		l = 1		l = 0, $(-2)^m$	

Table 4-1

Table of Multipliers

l \ m	0	1	2	3	4	5	6	7	8	9
0	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10
2	1	3	6	10	15	21	28	36	45	55
3	1	4	10	20	35	56	84	120	165	220
4	1	5	15	35	70	126	210	230	395	615

Table 4-2

Example IV

Example I in chapter II presents a characteristic equation

$$0.04s^4 + 0.34s^3 + (0.2\alpha+1.12)s^2 + (0.5\alpha+\beta+1.7)s + 2\beta + 1 = 0 \tag{4-9}$$

which has been shown to contain no singular lines. Suppose, for example, it is desired to have a singular line for the roots $\zeta = 0.5$, $\omega_n = 1$. If equation 4-9 represents the system in figure 4.1a, addition of a cascade compensator with transfer function

$$G_c(s) = \frac{b\alpha+c\beta+f}{2\beta+1} \tag{4-10}$$

results in a new characteristic equation

$$0.04s^4 + 0.34s^3 + (0.2\alpha+1.12)s^2 + (0.5\alpha+\beta+1.7)s + b\alpha + c\beta + f = 0, \tag{4-11}$$

where, if the system is made singular by definition,

$$B_1 = -b + 0.2 = B_2 = -0.3 \tag{4-12}$$

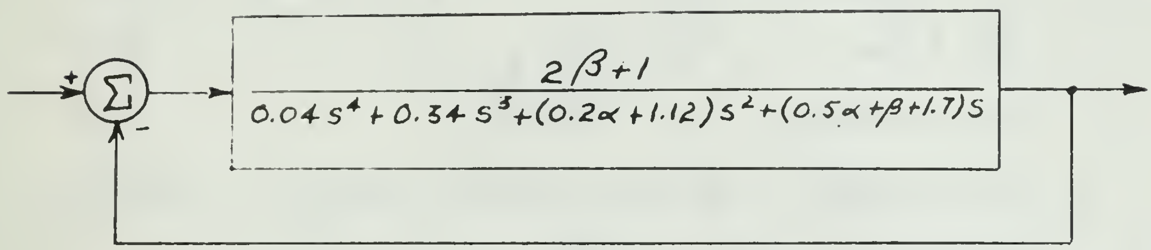
$$C_1 = -c = C_2 = -1 \tag{4-13}$$

$$F_1 = -f + 0.78 = F_2 = -0.62; \tag{4-14}$$

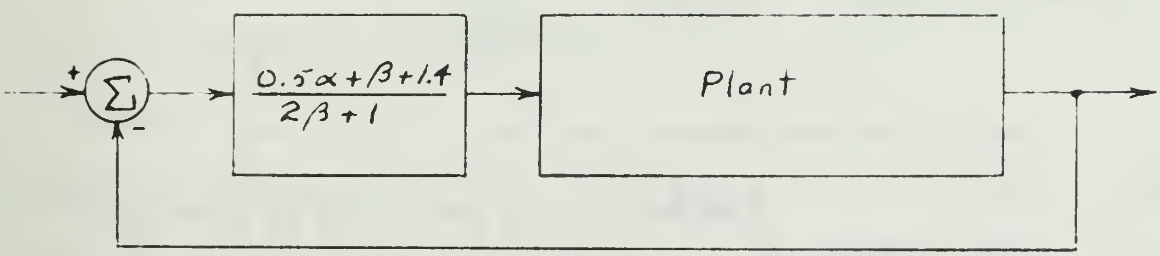
and the system then has a singular line solution for the roots $\zeta = 0.5$, $\omega_n = 1$ when $b = 0.5$, $c = 1$, and $f = 1.4$. The compensated system is then as shown in figure 4.1b; and the singular line for the desired roots is

$$0.3\alpha + \beta + 0.62 = 0. \tag{4-15}$$

Although, as stated in chapter two, a system may possess an infinite number of singular lines, one for a desired root pair may still not exist without compensation.



(a)



(b)

Fig. 4.1 Example IV-4th Order Plant

Example V

Figure 4.2a shows a system which was contrived to have singular lines. The characteristic equation is

$$s^3 + \alpha s^2 + (K_1+8K_1K_2\beta)s + 20K_1K_2\beta = 0, \qquad (4\text{-}16)$$

where

$$B_1 = \omega_n^2 \qquad B_2 = 2\zeta\omega_n^2 \qquad (4\text{-}17)$$

$$C_1 = -20K_1K_2 \qquad C_2 = -8K_1K_2\omega_n \qquad (4\text{-}18)$$

$$F_1 = -2\zeta\omega_n^3 \qquad F_2 = \omega_n^3 - 4\zeta^2\omega_n^3 - K_1\omega_n. \qquad (4\text{-}19)$$

By the definition for a singular line,

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = \begin{vmatrix} -F_1 & C_1 \\ -F_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & -F_1 \\ B_2 & -F_2 \end{vmatrix} = 0 \qquad (4\text{-}20)$$

Substituting,

$$\begin{vmatrix} \omega_n^2 & -20K_1K_2 \\ 2\zeta\omega_n^2 & -8K_1K_2 \end{vmatrix} = 40K_1K_2\zeta\omega_n^2 - 8K_1K_2\omega_n^2 = 0 \qquad (4\text{-}21)$$

$$\begin{vmatrix} 2\zeta\omega_n^3 & -20K_1K_2 \\ (4\zeta^2\omega_n^3-\omega_n^3+K_1\omega_n) & -8K_1K_2 \end{vmatrix} = 80K_1K_2\zeta^2\omega_n^3 - 20K_1K_2\omega_n^3 + 20K_1^2K_2\omega_n - 16K_1K_2\zeta\omega_n^3 = 0 \qquad (4\text{-}22)$$

$$\begin{vmatrix} \omega_n^2 & 2\zeta\omega_n^3 \\ 2\zeta\omega_n^2 & (4\zeta^2\omega_n^3-\omega_n^3+K_1\omega_n) \end{vmatrix} = 4\zeta^2\omega_n^5 - \omega_n^5 + K_1\omega_n^3 - 4\zeta^2\omega_n^3 = 0. \qquad (4\text{-}23)$$

Solving, equation 4-21 gives

$$\zeta = 0.2, \qquad (4\text{-}24)$$

and equations 4-22 and 4-23 give

$$\omega_n^2 = K_1. \qquad (4\text{-}25)$$

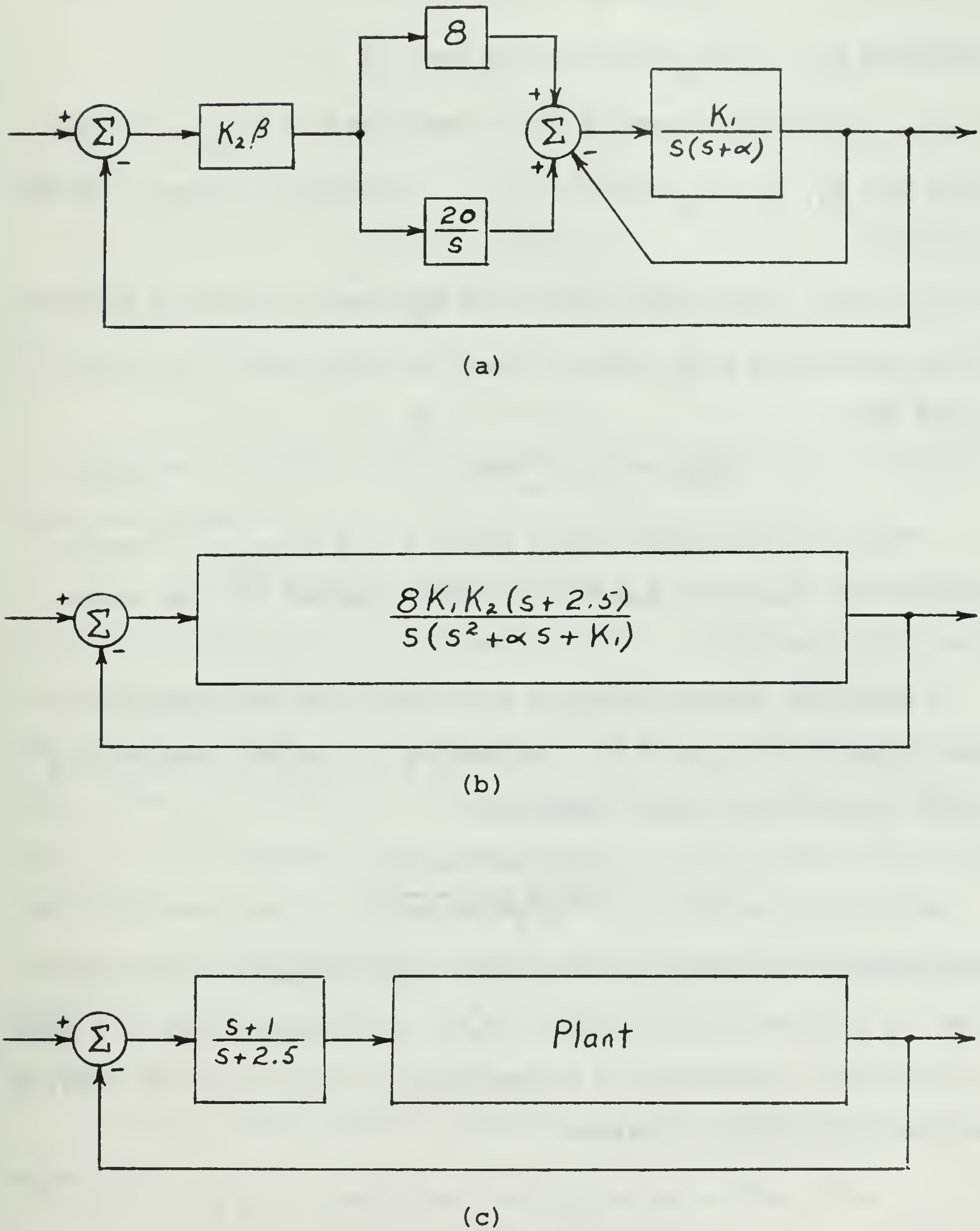


Fig. 4.2 Example V-3rd Order Plant

The singular line for roots $\zeta = 0.2$, $\omega_n = K_1$ is described by both

$$\alpha - 20K_2\beta - 0.4\sqrt{K_1} = 0 \qquad (4\text{-}26)$$

from the B_1, C_1, F_1 coefficients and by

$$0.4\alpha - 8K_2\sqrt{K_1}\beta - 0.16\sqrt{K_1} = 0 \qquad (4\text{-}27)$$

from the B_2, C_2, F_2 coefficients. Dividing equation 4-27 by 0.4,

$$\alpha - 20K_2\sqrt{K_1}\beta - 0.4\sqrt{K_1} = 0. \qquad (4\text{-}28)$$

Since equations 4-26 and 4-28 must be identical, it is obvious that

$$\sqrt{K_1} = K_1 = \omega_n^2 = \omega_n = 1. \qquad (4\text{-}29)$$

This is fortunate if the roots $\zeta = 0.2$, $\omega_n = 1$ are desired, but suppose a singular line is wanted for the roots $\zeta = 0.5$, $\omega_n = 1$.

By block diagram algebra the system can be reduced to that shown in figure 4.2b. Selecting a cascade compensator which cancels the plant numerator

$$G_c(s) = \frac{8K_1K_2\beta s+b\alpha+c\beta+f}{8K_1K_2\beta(s+2.5)} \qquad (4\text{-}30)$$

where the s^1 numerator coefficient must contain β in order that C_2 be non-zero and where $8K_1K_2$ is inserted with the idea of hopefully preserving a resemblance of the original gain, a new characteristic equation

$$s^3 + \alpha s^2 + (K_1+8K_1K_2\beta)s + b\alpha + c\beta + f = 0 \qquad (4\text{-}31)$$

is formed. For singularity

$$B_1 = -b+1 = B_2 = 1; \qquad b = 0 \tag{4-32}$$

$$C_1 = -c \quad = C_2 = -8K_1K_2; \quad c = 8K_1K_2 \tag{4-33}$$

$$F_1 = -f-1 = F_2 = -K_1; \qquad f = K_1-1. \tag{4-34}$$

If now $K_1 = K_2 = 1$, the compensator becomes the readily realizable

$$G_c(s) = \frac{s+1}{s+2.5}, \tag{4-35}$$

with the compensated system in figure 4.2c having the singular line

$$\alpha - 8\beta - 1 = 0. \tag{4-36}$$

Figure 4.3 shows the parameter plane plot of the characteristic equation

$$s^3 + \alpha s^2 + (8\beta+1)s + 8\beta = 0 \tag{4-37}$$

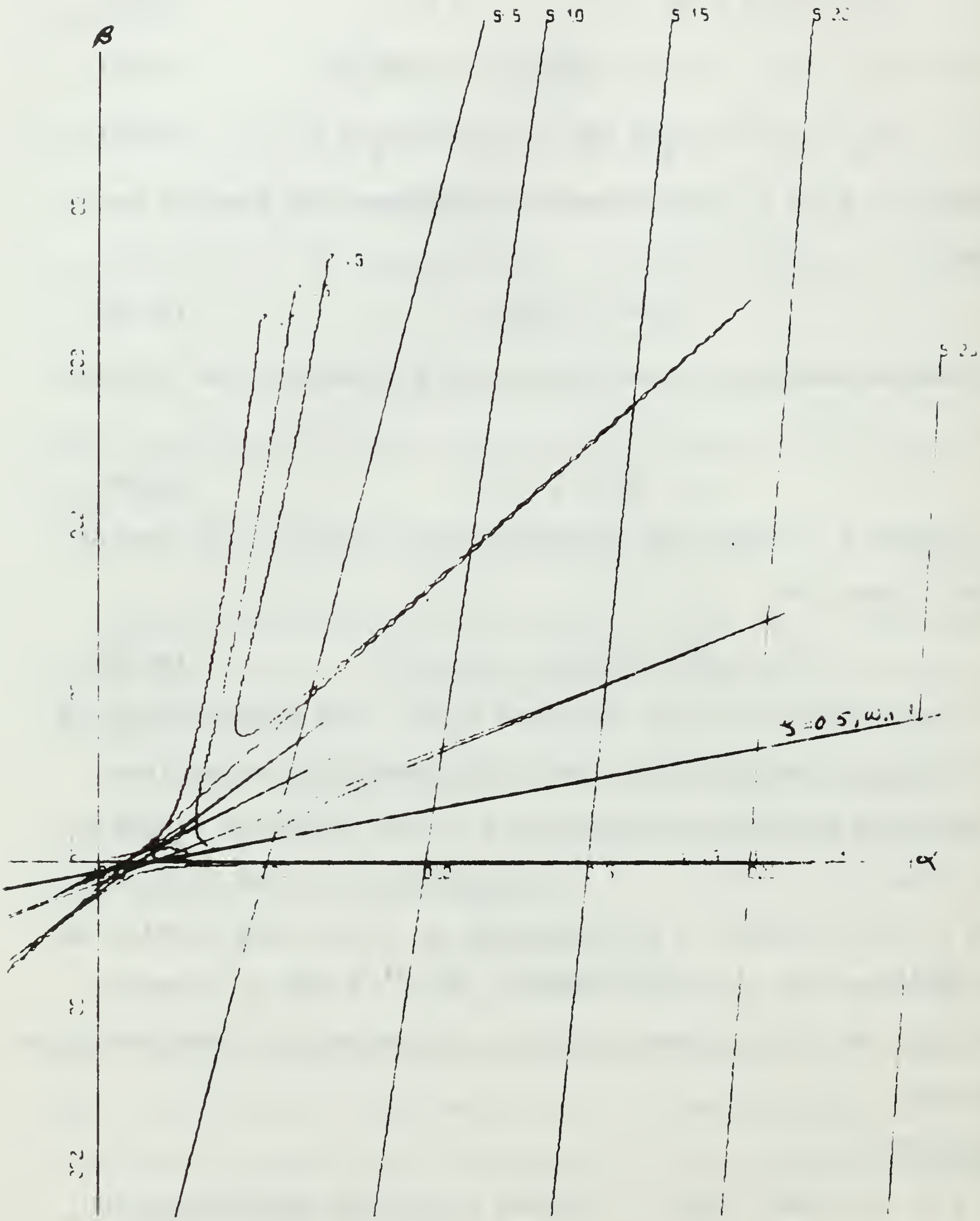
with the singular line of equation 4-36. The intersection of the singular line with the real root lines then determines the complete solution of equation 4-37 for selected singular line values of α and β. It is interesting to note that the plots of the constant ζ and constant ω_n lines form saddles as they approach the singular values. This is due to computer truncation as the program solution approaches the indeterminate condition.

<u>Example VI</u>

For the plant shown in figure 4.4a with characteristic equation

$$s^3 + (\alpha+2)s^2 + 2\alpha s + KK_1 = 0, \tag{4-38}$$

cascade compensation is desired to produce a singular line for roots $\zeta = 0.5$, $\omega_n = 1$. The first step is to cancel the



Scale: α=5/in.,β=1/in.

Fig. 4.3 Example V-Characteristic Equation

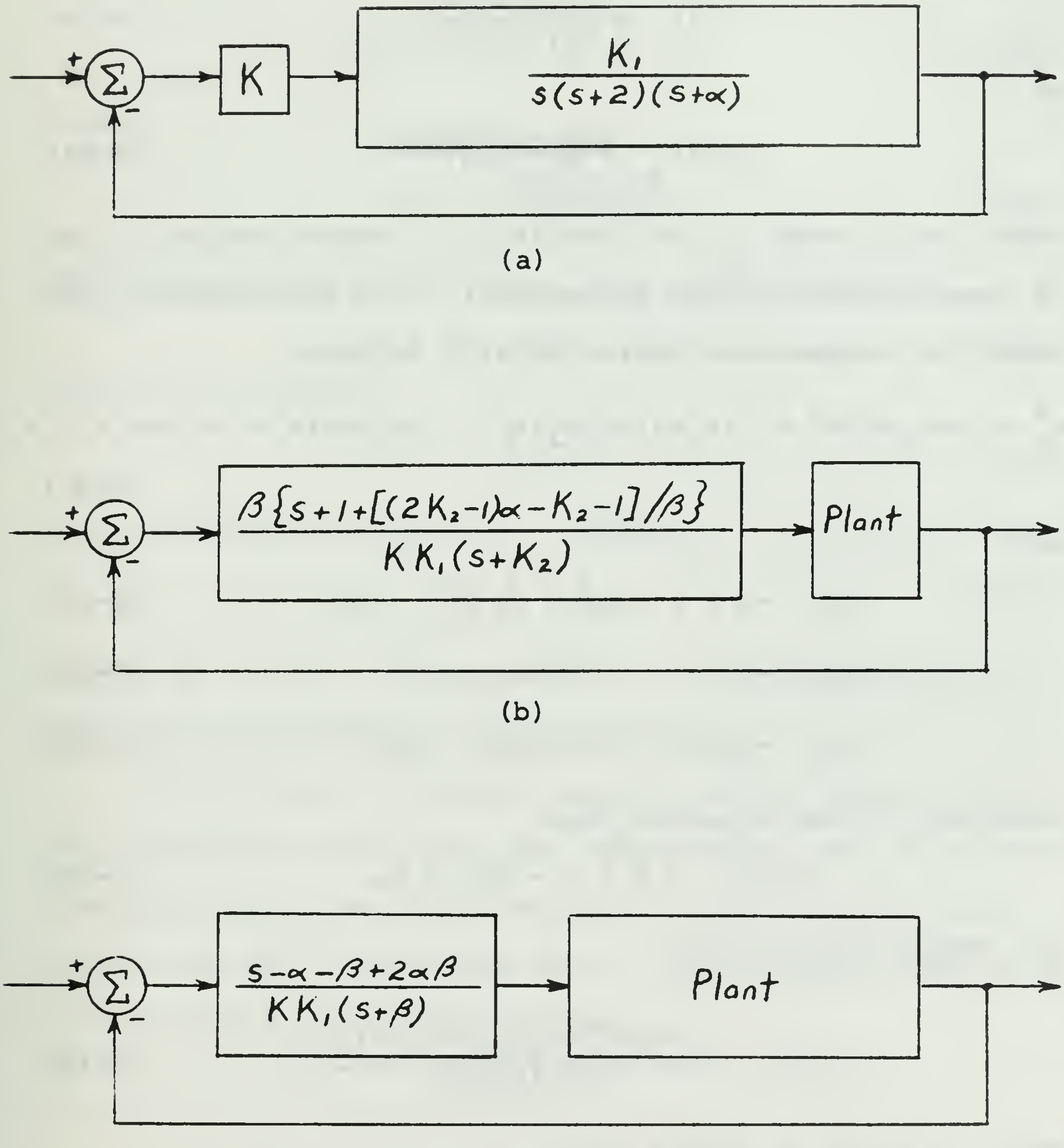


Fig. 4.4 Example VI-3rd Order Plant

plant numerator while concurrently introducing the second parameter in at least two different characteristic equation coefficients so that neither C_1 nor C_2 are zero. This can be accomplished in at least two ways:

$$G_c(s) = \frac{\beta s+b\alpha+c\beta+f}{KK_1(s+K_2)} \tag{4-39}$$

or

$$G_c(s) = \frac{s+b\alpha+c\beta+d\alpha\beta+f}{KK_1(s+\beta)}, \tag{4-40}$$

where the $\alpha\beta$ term in the numerator is needed because of the $\alpha\beta$ term produced by the denominator. The first method produces the compensated characteristic equation

$$s^4 + (\alpha+K_2+2)s^3 + [(K_2+2)\alpha+2K_2]s^2 + (2K_2\alpha+\beta)s + b\alpha + c\beta + f = 0, \tag{4-41}$$

with

$$B_1 = -b + 1 + K_2 = B_2 = 2 - K_2 \tag{4-42}$$

$$C_1 = -c = C_2 = -1 \tag{4-43}$$

$$F_1 = -f + K_2 - 2 = F_2 = 2K_2 - 1, \tag{4-44}$$

resulting in the singular line

$$(2-K_2)\alpha + \beta + 1 - 2K_2 = 0. \tag{4-45}$$

The compensator is then

$$G_c(s) = \frac{\beta\{s+1+[(2K_2-1)\alpha-K_2-1]/\beta\}}{KK_1(s+K_2)}, \tag{4-46}$$

which is shown in figure 4.4b.

With the second method the characteristic equation becomes

$$s^4 + (\alpha+\beta+2)s^3 + (2\alpha+2\beta+\alpha\beta)s^2 + (2\alpha\beta+1)s + b\alpha + c\beta + d\alpha\beta + f = 0, \tag{4-47}$$

where

$$B_1 = -b+1 = B_2 = 2 \tag{4-48}$$

$$C_1 = -c+1 = C_2 = 2 \tag{4-49}$$

$$D_1 = -d+1 = D_2 = -1 \tag{4-50}$$

$$F_1 = -f-2 = F_2 = -2, \tag{4-51}$$

making

$$G_c(s) = \frac{s-\alpha-\beta+2\alpha\beta}{KK_1(s+\beta)}\ , \tag{4-52}$$

which is shown in figure 4.4c. The singular line is

$$2\alpha + 2\beta - \alpha\beta - 2 = 0, \tag{4-53}$$

which may be written

$$\beta = \frac{6}{\alpha-2} + 2 \tag{4-54}$$

and is plotted in figure 4.5. Since

$$\lim_{\alpha\to\infty} \beta = 2\ , \tag{4-55}$$

β could be set to 2 to give singular line performance for small variations about a large value of α.

The preceding has demonstrated a rather simple mathematical approach to singular line compensation, but it is sufficient both for a practical solution to the singular line problem and for an appreciation of a more elegant method which is the subject of the next chapter.

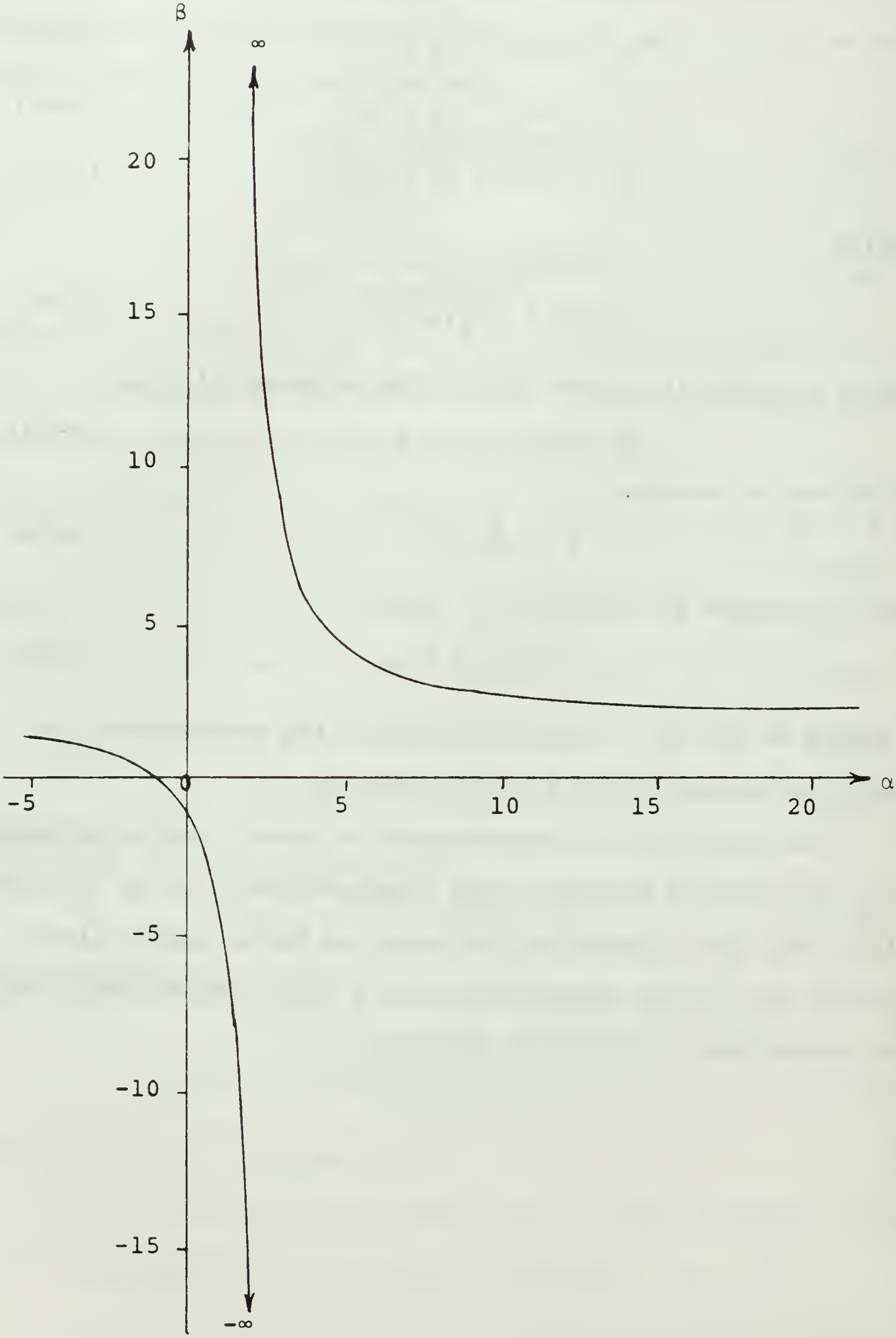


Fig. 4.5 Example VI-Non-linear Singular Line

CHAPTER V

COMPENSATING TECHNIQUES

Consider the uncompensated two parameter control system described by

$$B_1\alpha + C_1\beta + F_1 = 0 \qquad (5\text{-}1)$$

$$B_2\alpha + C_2\beta + F_2 = 0. \qquad (5\text{-}2)$$

If this system is now compensated by some function of s and γ, where γ is a function of α and β, equations 5-1 and 5-2 become

$$B_1\alpha + C_1\beta + D_1\gamma(\alpha,\beta) + F_1 = 0 \qquad (5\text{-}3)$$

$$B_2\alpha + C_2\beta + D_2\gamma(\alpha,\beta) + F_2 = 0 \qquad (5\text{-}4)$$

for the compensated system. After equation 5-4 is multiplied by D_1/D_2, the set becomes

$$B_1\alpha + C_1\beta + D_1\gamma(\alpha,\beta) + F_1 = 0 \qquad (5\text{-}5)$$

$$\frac{D_1B_2}{D_2}\alpha + \frac{D_1C_2}{D_2}\beta + D_1\gamma(\alpha,\beta) + \frac{D_1F_2}{D_2} = 0. \qquad (5\text{-}6)$$

Adding equations 5-5 and 5-6 gives

$$(B_1 + \frac{D_1B_2}{D_2})\alpha + (C_1 + \frac{D_1C_2}{D_2})\beta + 2D_1\gamma(\alpha,\beta) + F_1 + \frac{D_1F_2}{D_2} = 0, \qquad (5\text{-}7)$$

which, when solved for $\gamma(\alpha,\beta)$, yields

$$\gamma(\alpha,\beta) = -\frac{1}{2}[(\frac{B_1}{D_1} + \frac{B_2}{D_2})\alpha + (\frac{C_1}{D_1} + \frac{C_2}{D_2})\beta + \frac{F_1}{D_1} + \frac{F_2}{D_2}], \qquad (5\text{-}8)$$

the value of γ for the compensator. Also,

$$D_1\gamma(\alpha,\beta) = \left(\frac{-D_2B_1-D_1B_2}{2D_2}\right)\alpha + \left(\frac{-D_2D_1-D_1C_2}{2D_2}\right)\beta - \frac{D_2F_1}{2D_2} - \frac{D_1F_2}{2D_2}\,, \tag{5-9}$$

which, when substituted into equations 5-5 and 5-6, gives

$$\left(B_1 - \frac{D_2B_1}{2D_2} - \frac{D_1B_2}{2D_2}\right)\alpha + \left(C_1 - \frac{D_2C_1}{2D_2} - \frac{D_1C_2}{2D_2}\right)\beta + F_1 - \frac{D_2F_1}{2D_2} - \frac{D_1F_2}{2D_2} = 0 \tag{5-10}$$

$$\left(\frac{D_1B_2}{D_2} - \frac{D_2B_1}{2D_2} - \frac{D_1B_2}{2D_2}\right)\alpha + \left(\frac{D_1C_2}{D_2} - \frac{D_2C_1}{2D_2} - \frac{D_1C_2}{2D_2}\right)\beta + \frac{D_1F_2}{D_2} - \frac{D_2F_1}{2D_2} - \frac{D_1F_2}{2D_2} = 0. \tag{5-11}$$

Reducing, equations 5-10 and 5-11 become

$$(D_2B_1-D_1B_2)\alpha + (D_2C_1-D_1C_2)\beta + D_2F_1 - D_1F_2 = 0 \tag{5-12}$$

$$(D_1B_2-D_2B_1)\alpha + (D_1D_2-D_2C_1)\beta + D_1F_2 - D_2F_1 = 0, \tag{5-13}$$

which, differing by a constant factor of (-1), are obviously singular and are indeed the equation of a singular line for the compensated system.

Several examples utilizing equation 5-8 to design a compensator to produce a singular line described by equations 5-12 and 5-13 will be given for a variety of systems.

Example VII

Type 0-3rd Order-Feedback Compensation-One Parameter in Plant Denominator

For the system shown in figure 5.1a it is desired to obtain feedback compensation to produce a singular line for the roots $\zeta = 0.5$, $\omega_n = 1$. The first step is obviously to introduce the

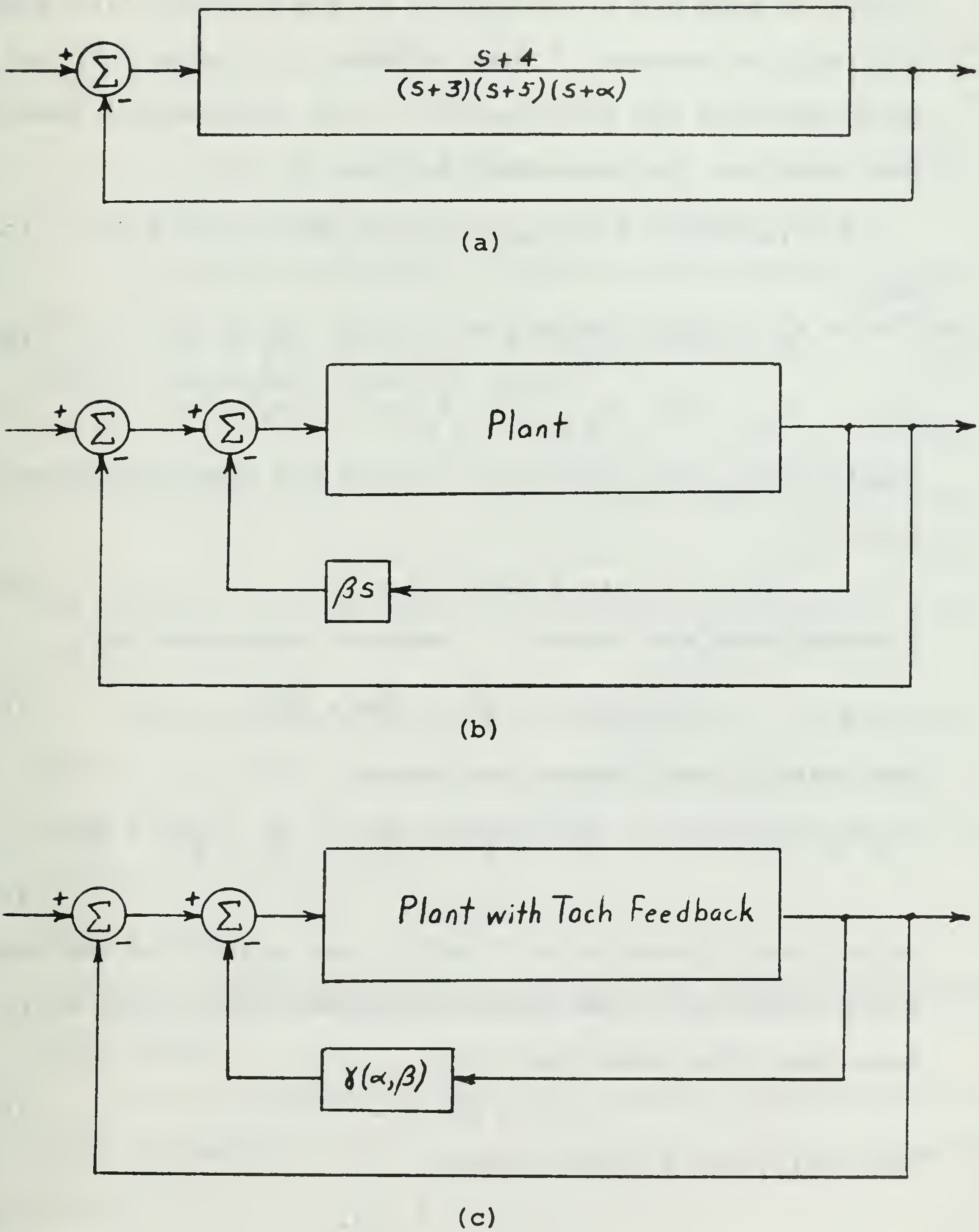


Fig. 5.1 Example VII-Type 0, 3rd Order Plant

second parameter. A slight amount of caution must be exercised to insure that the second parameter occurs in coefficients other than those which will contain the compensating function $\gamma(\alpha,\beta)$ so that the β coefficient in the singular line equation 5-12 will be non-zero. This is shown in figures 5.1b and 5.1c which features the introduction of the compensating function. The resultant characteristic equation is then

$$s^3 + (\alpha+\beta+8)s^2 + (8\alpha+4\beta+\gamma+16)s + 15\alpha + 4\gamma + 4 = 0, \qquad (5\text{-}14)$$

with

$$B_1 = -14 \quad C_1 = 1 \quad D_1 = -4 \quad F_1 = 3 \qquad (5\text{-}15)$$

$$B_2 = -7 \quad C_2 = -3 \quad D_2 = -1 \quad F_2 = -8 \,. \qquad (5\text{-}16)$$

Substitution into equations 5-12 and 5-8 gives the singular line

$$14\alpha + 13\beta + 35 = 0, \qquad (5\text{-}17)$$

produced when the system is feedback compensated by

$$\gamma(\alpha,\beta) = -\frac{21}{4}\alpha - \frac{11}{8}\beta - \frac{29}{8} \,. \qquad (5\text{-}18)$$

The characteristic equation is then

$$s^3 + (\alpha+\beta+8)s^2 + (\frac{11}{4}\alpha + \frac{21}{8}\beta + \frac{99}{8})s - 6\alpha - \frac{11}{2}\beta - \frac{21}{2} = 0, \qquad (5\text{-}19)$$

which, when divided by $s^2 + s + 1$, the product of the constant roots, results in the reduced characteristic equation, or in this case, the remaining root,

$$s + \alpha + \beta + 7 = 0. \qquad (5\text{-}20)$$

Obviously, for a stable system,

$$-\alpha - \beta - 7 \leq 0. \qquad (5\text{-}21)$$

Since, from equation 5-17,

$$\beta = -\frac{14}{13}\alpha - \frac{35}{13}, \tag{5-22}$$

$$\frac{1}{13}\alpha - \frac{56}{13} \leq 0, \tag{5-23}$$

and

$$\alpha \leq 56 \tag{5-24}$$

for a stable system operated on the singular line.

If Routh's stability criterion is applied to equation 5-19, a stability area in the parameter plane is established by the boundaries formed by curves

$$\alpha + \beta + 8 = 0 \tag{5-25}$$

$$22\alpha^2 + 43\alpha\beta + 227\alpha + 21\beta^2 + 223\beta + 708 = 0 \tag{5-26}$$

$$12\alpha + 11\beta + 21 = 0. \tag{5-27}$$

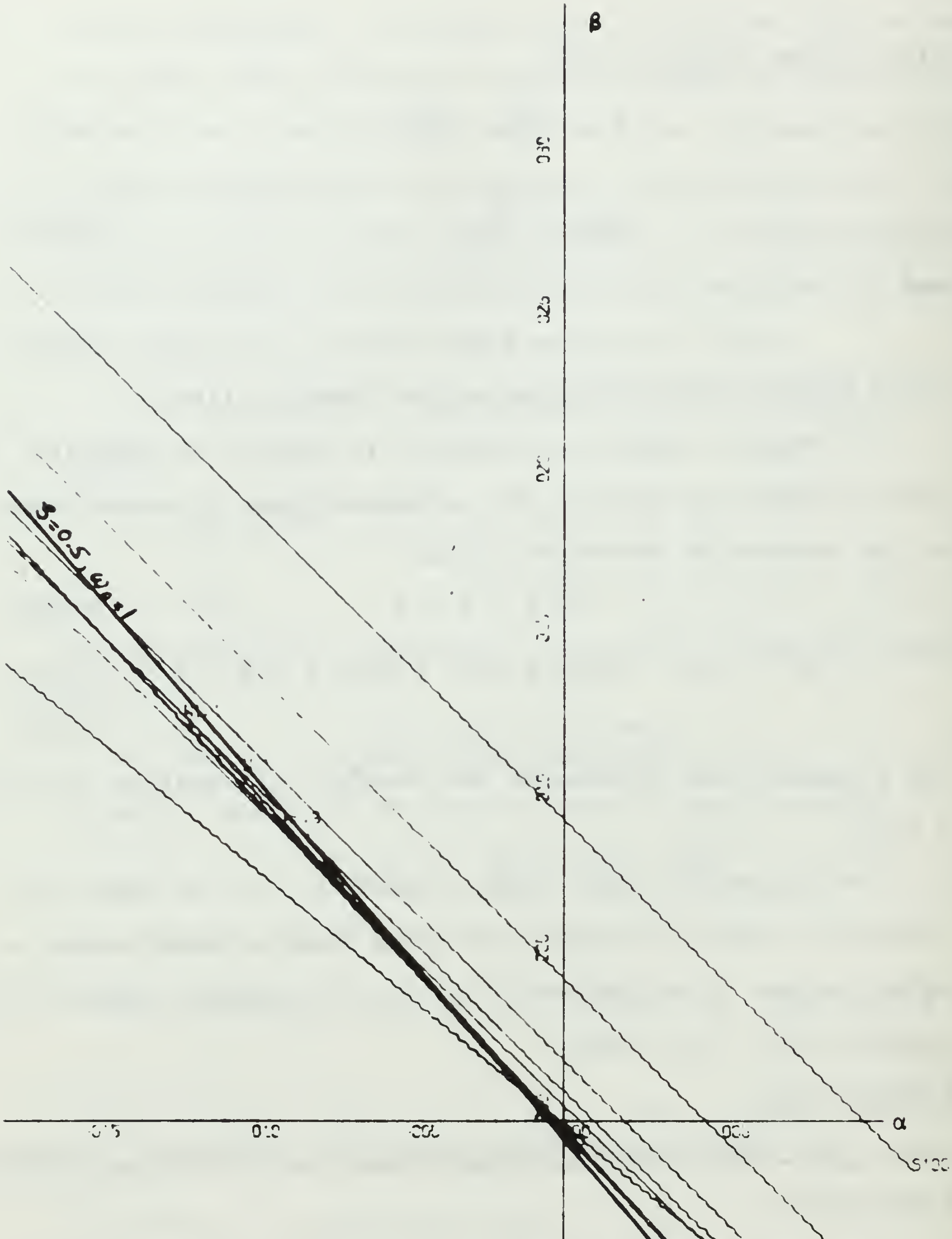
The singular line intersects the boundary of equation 5-27 at $\alpha = 56$.

The parameter plane plot of equation 5-19 is shown in figure 5.2, where the real root lines have a finite error in slope because an unresolved "bug" in the computer program for graphing real root lines.

Example VIII

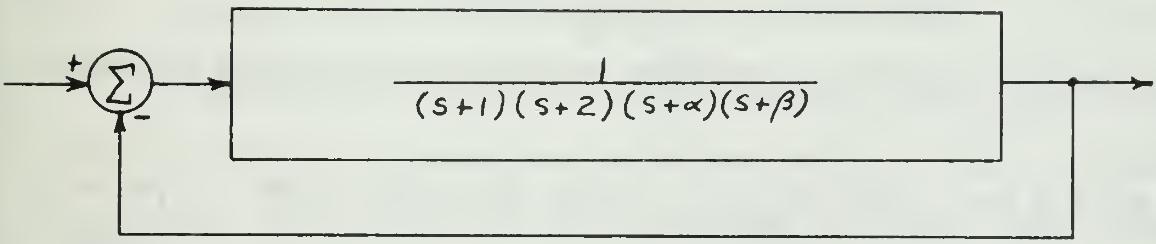
Type 0-4th Order-Cascade Compensation-Two Parameters in Plant Denominator

For the system in figure 5.3a, cascade compensation is added as shown in figure 5.3b, resulting in the characteristic equation

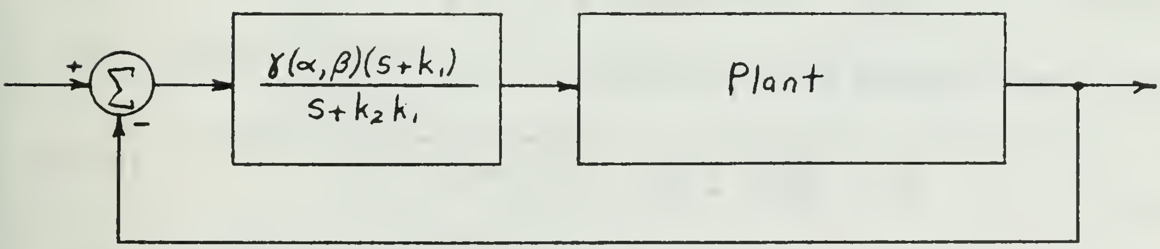


Scale:α=50/in.,β=50/in.

Fig. 5.2 Example VII-Characteristic Equation



(a)



(b)

Fig. 5.3 Example VIII-Type 0, 4th Order Plant

$$s^5 + (\alpha+\beta+k_2k_1+3)s^4 + [(k_2k_1+3)(\alpha+\beta) + \alpha\beta + 3k_1k_2]s^3$$

$$+ [(3k_2k_1+2)(\alpha+\beta) + (k_2k_1+3)\alpha\beta]s^2 + [2k_2k_1(\alpha+\beta) \qquad (5\text{-}28)$$

$$+ (3k_2k_1+2)\alpha\beta+\gamma]s + 2k_2k_1\alpha\beta + k_1\gamma = 0 \ ,$$

giving, for roots $\gamma = 0.5$, $\omega_n = 1$,

$$B_1 = C_1 = 2k_2k_1-1 \quad D_1 = -k_2k_1+2 \quad E_1 = -k_1 \quad F_1 = -3k_2k_2+1 \qquad (5\text{-}29)$$

$$B_2 = C_2 = k_2k_1+1 \quad D_2 = -2k_2k_1+1 \quad E_2 = -1 \quad F_2 = -k_2k_1-2. \qquad (5\text{-}30)$$

For the case of the $\alpha\beta$ product, equation 5-12 can be extended to

$$(E_2B_1-E_1B_2)\alpha + (E_2C_1-E_1C_2)\beta + (E_2D_1-E_1D_2)\alpha\beta + E_2F_1 - E_1F_2 = 0, \qquad (5\text{-}31)$$

which leads to the singular line

$$(1-2k_2k_1+k_2k_1^2+k_1)(\alpha+\beta) + (k_2k_1-2-2k_2k_1^2+k_1)\alpha\beta$$

$$+ 3k_2k_1 - 1 - k_2k_1^2 - 2k_1 = 0. \qquad (5\text{-}32)$$

For a linear singular line,

$$k_2k_1 - 2 - 2k_2k_1^2 + k_1 = 0, \qquad (5\text{-}33)$$

which, when solved for k_1, yields

$$k_1 = \frac{k_2+1}{4k_2} \pm \frac{1}{4k_2}\sqrt{k_2^2-14k_2+1} \ . \qquad (5\text{-}34)$$

Setting the radical equal to zero and solving for k_2,

$$13.93 \leq k_2 \leq 0.07 \qquad (5\text{-}35)$$

for real values of k_1. Choosing $k_2 = 13.93$ in order to obtain the greatest possible residue for the singular roots to increase their dominance makes $k_1 = 0.268$. The singular line for the compensated system is then

$$\alpha + \beta - 1.67 = 0. \tag{5-36}$$

For the compensator, equation 5-8 can be extended to

$$\gamma(\alpha,\beta) = -\frac{1}{2}\left[\left(\frac{B_1}{E_1}+\frac{B_2}{E_2}\right)\alpha + \left(\frac{C_1}{E_1}+\frac{C_2}{E_2}\right)\beta + \left(\frac{D_1}{E_1}+\frac{D_2}{E_2}\right)\alpha\beta + \frac{F_1}{E_1}+\frac{F_2}{E_2}\right], \tag{5-37}$$

which for $k_1 = 0.268$ and $k_2 = 13.93$ gives

$$G_c(s) = \frac{[14.47(\alpha+\beta)-6.49\alpha\beta-21.9](s+0.268)}{s+3.73} \tag{5-38}$$

Although the pole-zero configuration of the above compensating filter is recognized as undesirable in that it spans more than one decade, it was chosen only to keep the singular line linear for illustrative purposes. A more physically feasible filter will result in a non-linear singular curve.

The resultant characteristic equation then becomes

$$s^5 + (\alpha+\beta+6.73)s^4 + [6.73(\alpha+\beta)+\alpha\beta+11.2]s^3 + [13.2(\alpha+\beta)+6.73\alpha\beta]s^2$$
$$+ [21.94(\alpha+\beta)+6.71\alpha\beta-21.9]s + 3.87(\alpha+\beta) + 5.73\alpha\beta - 5.87 = 0. \tag{5-39}$$

The reduced characteristic equation is then

$$s^3 + (\alpha+\beta+5.73)s^2 + [5.73(\alpha+\beta)+\alpha\beta+4.47]s + 6.47(\alpha+\beta)$$
$$+ 5.73\alpha\beta - 10.2 = 0. \tag{5-40}$$

Since, from equation 5-36,

$$\beta = -\alpha + 1.67, \tag{5-41}$$

equation 5-40 becomes

$$s^3 + 7.4s^2 + (-\alpha^2+1.67\alpha+14.13)s - 5.73\alpha^2 + 14.13\alpha + 0.6 = 0, \tag{5-42}$$

for which a Routh's stability analysis yields a range of operation

$$-0.04 \leq \alpha \leq 2.51. \qquad (5\text{-}43)$$

Figure 5.4 shows the parameter plane plot of the reduced characteristic equation. For clarity figures 5.5 and 5.6 show only the lines $\zeta = 0.8$ and $\zeta = 0.9$, respectively. From these plots it is clear that the second pair of complex roots shift continuously for extremely small changes in α and β.

Figure 5.7 is the parameter plane plot of characteristic equation 5-39.

Example IX

Type 1-2nd Order-Feedback Compensation-No Parameters in Plant

For the uncompensated system of figure 5.8a, the two parameters must be introduced as in figure 5.8b. Tachometer feedback is used in order to retain the system type. Next the singular line compensating function is introduced with accelerometer feedback so that the singular line coefficients will be non-zero. The characteristic equation is then

$$(10\gamma+1)s^2 + (10\alpha+10\beta+1)s + 10 = 0, \qquad (5\text{-}44)$$

with, for $\zeta = 0.5$, $\omega_n = 1$,

$$B_1 = C_1 = 0 \quad D_1 = 10 \quad F_1 = -9 \qquad (5\text{-}45)$$

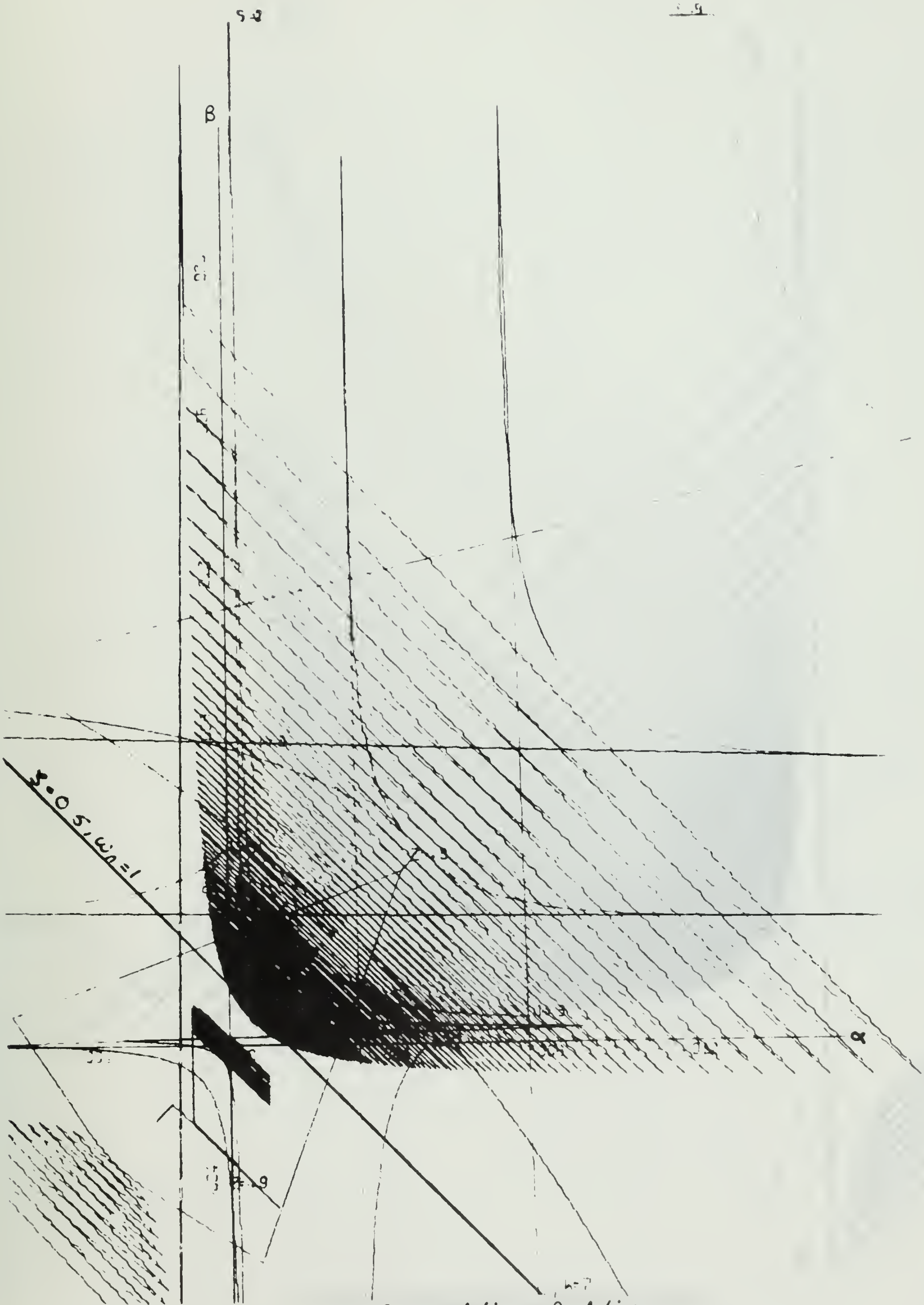
$$B_2 = C_2 = -10 \quad D_2 = 10 \quad F_2 = 0\ . \qquad (5\text{-}46)$$

This produces a singular line

$$10\alpha + 10\beta - 9 = 0, \qquad (5\text{-}47)$$

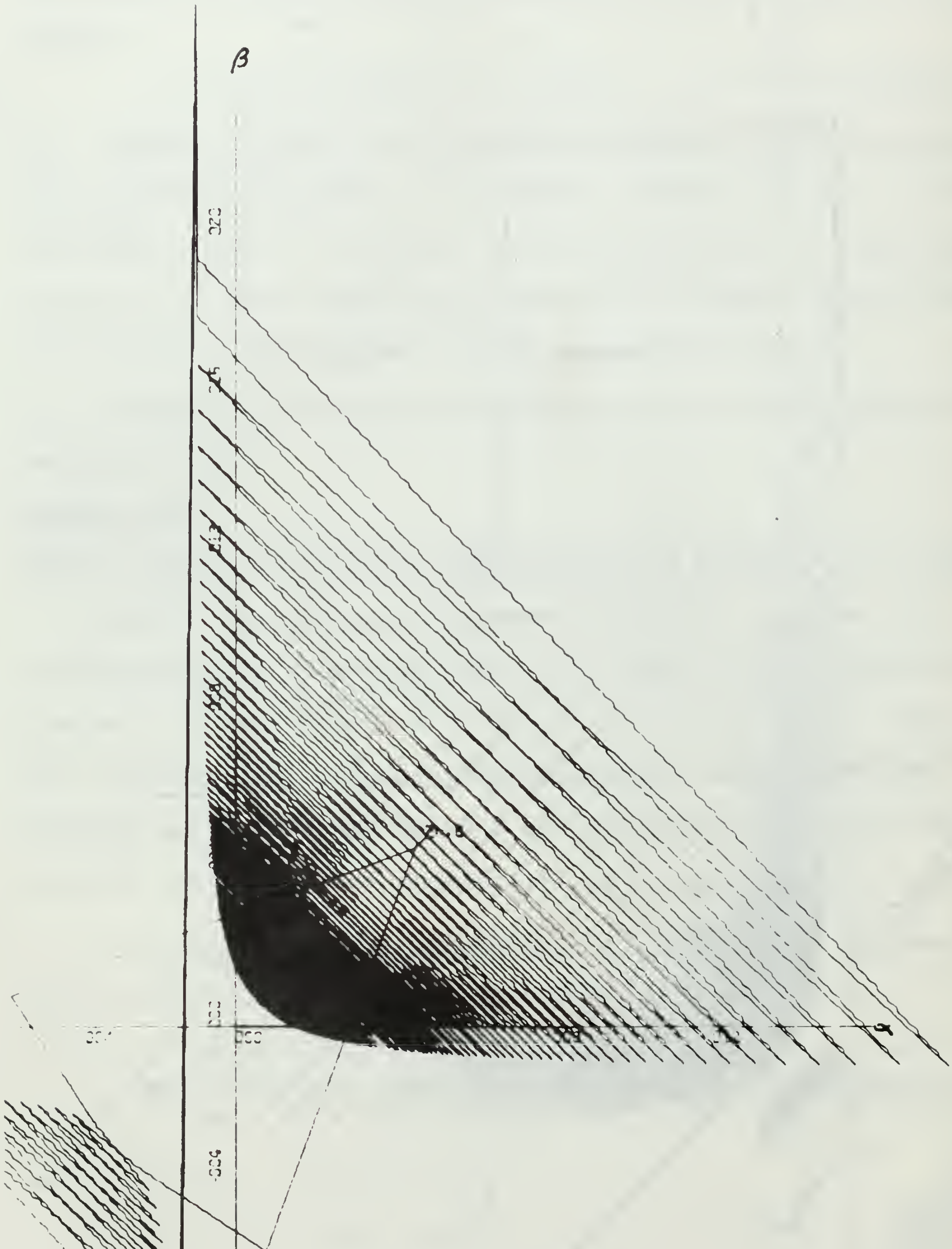
and

$$\gamma(\alpha,\beta) = \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{9}{20}\ . \qquad (5\text{-}48)$$



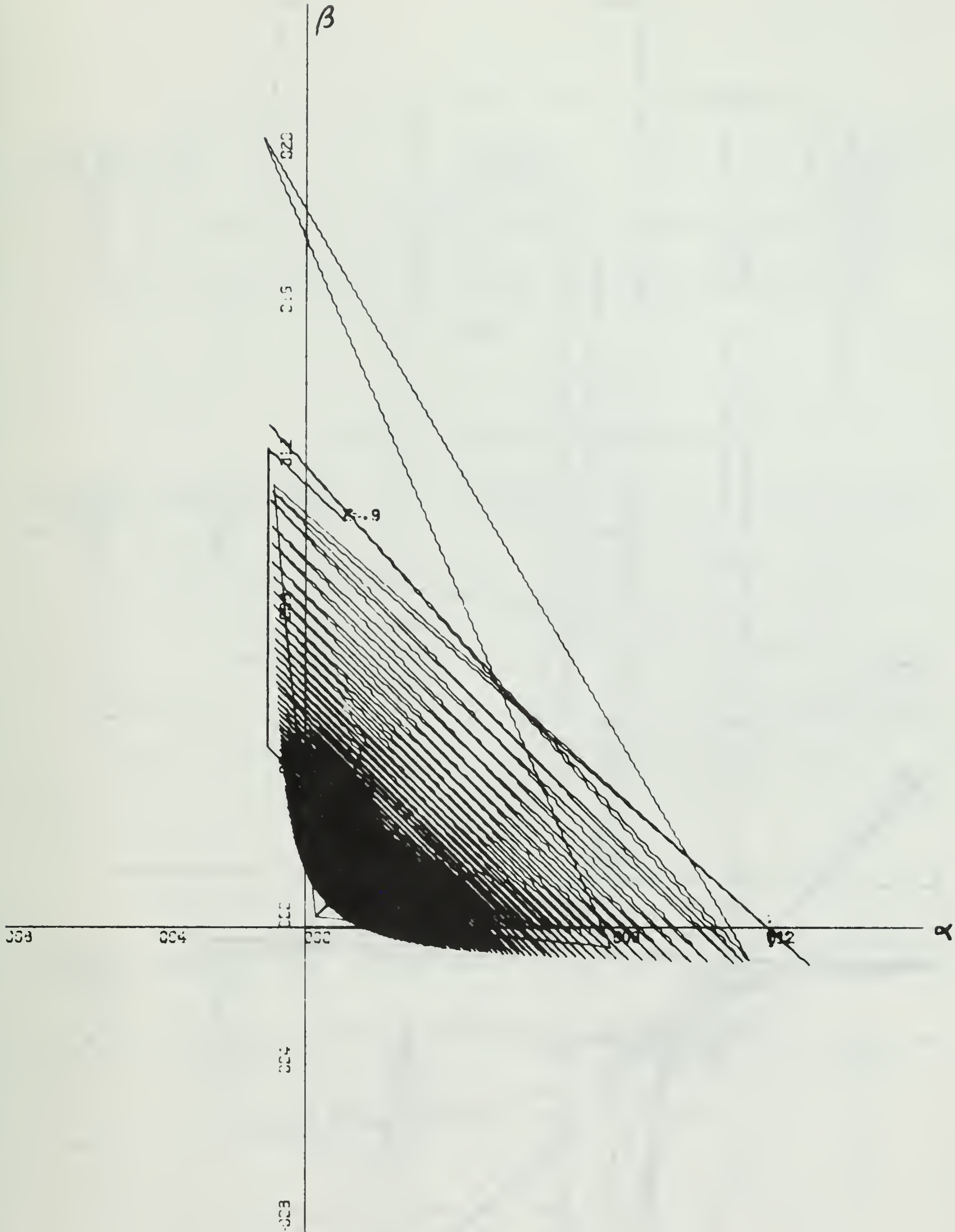
Scale:α=4/in.,β=4/in.

Fig. 5.4 Example VIII-Reduced Characteristic Equation



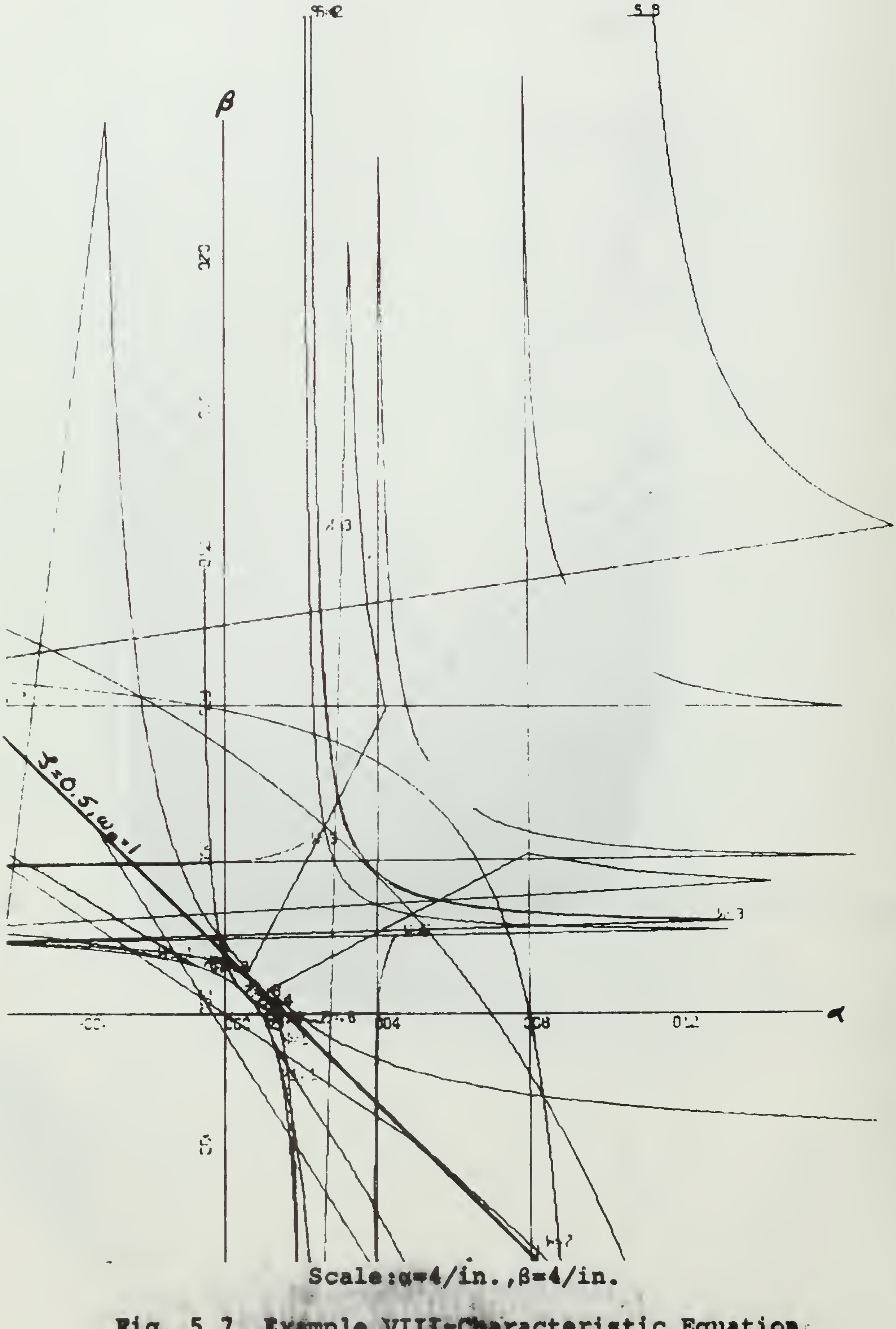
Scale:α=4/in.,β=4/in.

Fig. 5.5 Example VIII-Zeta = 0.8



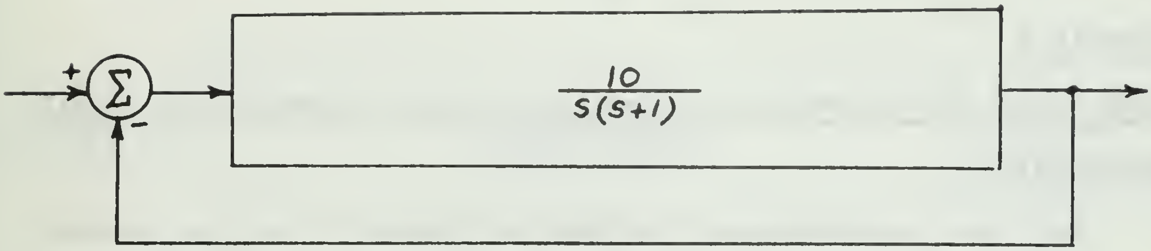
Scale:α=4/in.,β=4/in.

Fig. 5.6 Example VIII-Zeta = 0.9

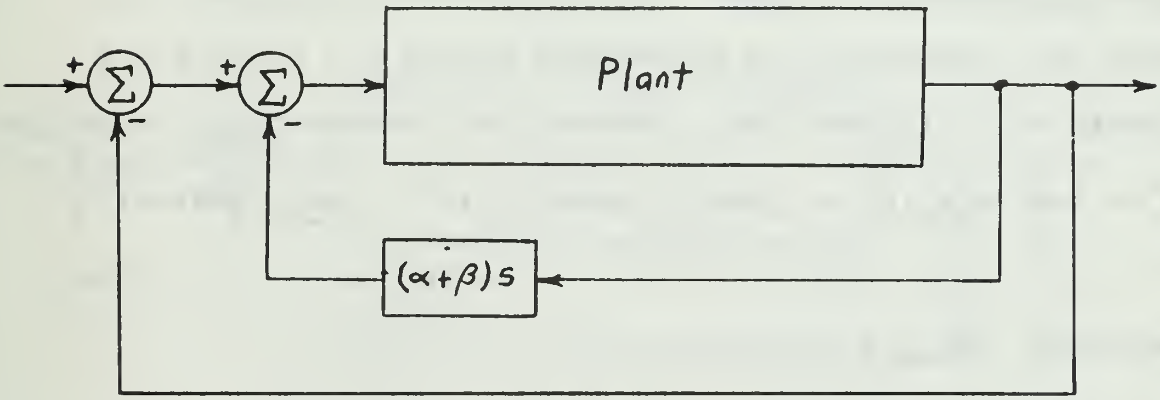


Scale: α=4/in., β=4/in.

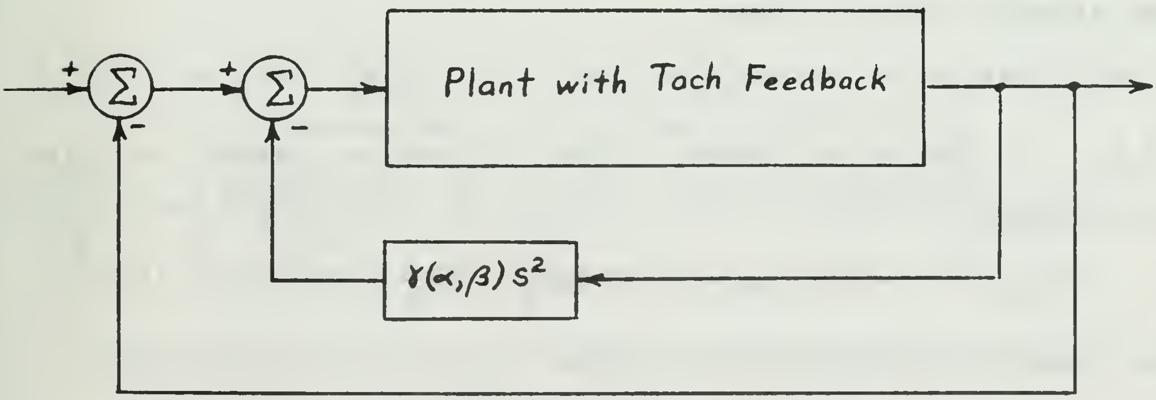
Fig. 5.7 Example VIII-Characteristic Equation



(a)



(b)



(c)

Fig. 5.8 Example IX-Type 1, 2nd Order Plant

By substitution of the singular line relationship into equation 5-48,

$$\gamma = \frac{9}{10} \ . \qquad (5\text{-}49)$$

<u>Example X</u>

<u>Type 1-3rd Order-Cascade Compensation-One Parameter in Plant Denominator</u>

For the uncompensated system of figure 5.9a, the second parameter is introduced via tachometer feedback both to preserve the system type and to avoid a $\beta\gamma$ product when the cascade compensator is added. This is shown in figure 5.9b. Then the compensator is introduced at the $k = 0$ and $k = 1$ levels as in figure 5.9c, yielding the characteristic equation

$$s^4 + (\alpha+1+k_2k_1)s^3 + [(1+k_2k_1)\alpha+\beta+k_2k_1]s^2 + [k_2k_1(\alpha+\beta)+4\gamma]s + 4k_1\gamma = 0. \qquad (5\text{-}50)$$

As usual, for $\zeta = 0.5$, $\omega_n = 1$,

$$B_1 = k_2k_1 \quad C_1 = 1 \quad D_1 = -4k_2 \quad F_1 = -1 \qquad (5\text{-}51)$$

$$B_2 = 1 \quad C_2 = -k_2k_1+1 \quad D_2 = -4 \quad F_2 = k_2k_1-1. \qquad (5\text{-}52)$$

The singular line is then

$$(-4k_2k_1+4k_1)\alpha + (-4-4k_2k_1^2+4k_1)\beta + 4 + 4k_2k_1^2 - 4k_1 = 0. \qquad (5\text{-}53)$$

If $k_2 = 10$ for a one decade filter pole-zero spread, equation 5-53 becomes

$$-36k_1\alpha + (-40k_1^2+4k_1-4)\beta + 40k_1^2 - 4k_1 + 4 = 0. \qquad (5\text{-}54)$$

The compensating function is then

$$\gamma(\alpha,\beta) = \frac{1}{8k_1}[11k_1\alpha+(1+k_1-10k_1^2)\beta-1+10k_1^2-k_1], \qquad (5\text{-}55)$$

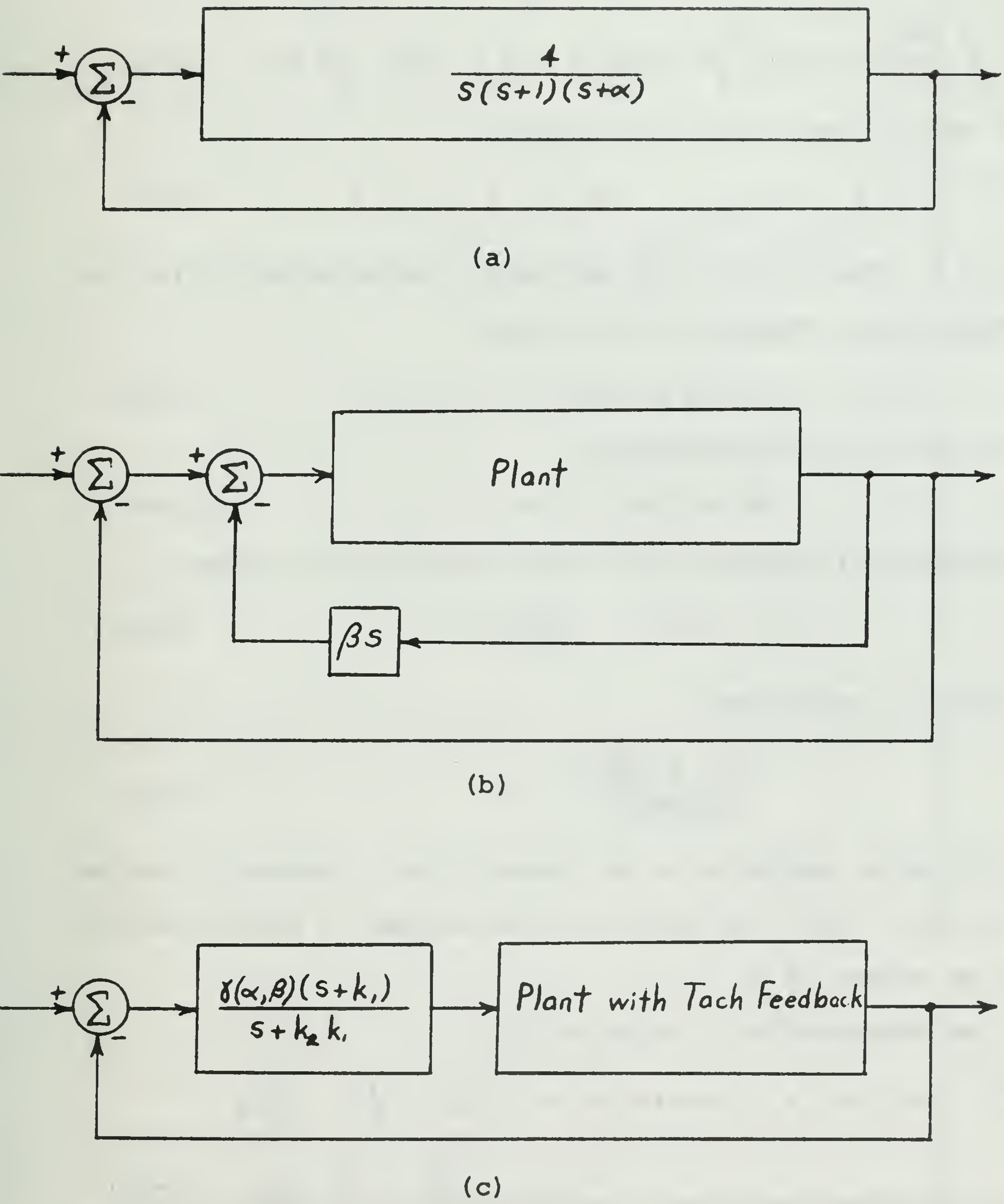


Fig. 5.9 Example X-Type 1, 3rd Order Plant

which gives a characteristic equation

$$s^4 + (\alpha+1+10k_1)s^3 + [(1+10k_1)\alpha+\beta+10k_1]s^2$$

$$+ [(10k_1 + \frac{11}{2})\alpha + (5k_1 + \frac{1}{2k_1} + \frac{1}{2})\beta - \frac{1}{2k_1} + 5k_1 - \frac{1}{2}]s$$

$$+ \frac{11k_1}{2}\alpha + (\frac{1}{2} + \frac{k_1}{2} - 5k_1^2)\beta - \frac{1}{2} + 5k_1^2 - \frac{k_1}{2} = 0 \qquad (5\text{-}56)$$

and a reduced characteristic equation

$$s^2 + (\alpha+10k_1)s + 10k_1\alpha + \beta - 1 = 0. \qquad (5\text{-}57)$$

If $k_1 = 2$, which gives good dominance characteristics to the singular roots, equation 5-57 becomes

$$s^2 + (\alpha+20)s + 20\alpha + \beta - 1 = 0; \qquad (5\text{-}58)$$

and the singular line becomes

$$6\alpha + 13\beta - 13 = 0. \qquad (5\text{-}59)$$

Substitution of equation 5-59 into equation 5-58 gives

$$s^2 + (\alpha+20)s + \frac{254}{13}\alpha = 0, \qquad (5\text{-}60)$$

which may be rewritten

$$\frac{\alpha(s + \frac{254}{13})}{s(s+20)} = -1 \qquad (5\text{-}61)$$

for root locus analysis as in figure 5.10. Obviously the two non-singular roots are real, and the system is stable for all positive values of α.

The characteristic equation is

$$s^4 + (\alpha+21)s^3 + (21\alpha+\beta+20)s^2 + (\frac{51}{2}\alpha + \frac{43}{4}\beta + \frac{37}{4})s$$

$$+ 11\alpha - \frac{37}{4}\beta + \frac{37}{2} = 0 \qquad (5\text{-}62)$$

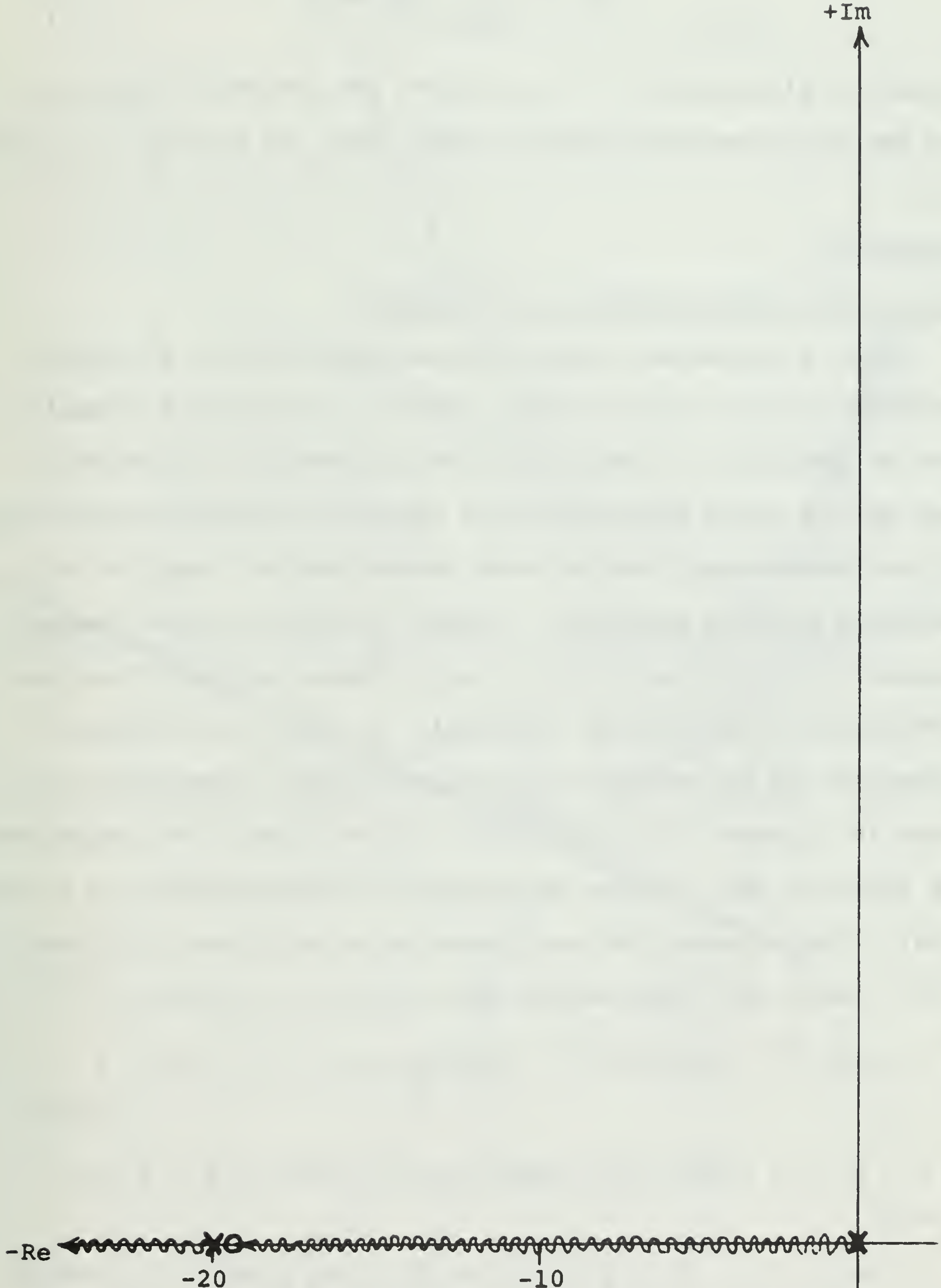


Fig. 5.10 Example X-Reduced Characteristic Equation Root Locus Plot

for the system compensated by

$$G_c(s) = \frac{(\frac{11}{8}\alpha - \frac{37}{16}\beta + \frac{37}{16})(s+2)}{s+20} . \qquad (5\text{-}63)$$

Parameter plane plots of the reduced characteristic equation and the characteristic equation are shown in figures 5.11 and 5.12.

Example XI

Type 1-4th Order-Parameter in Numerator

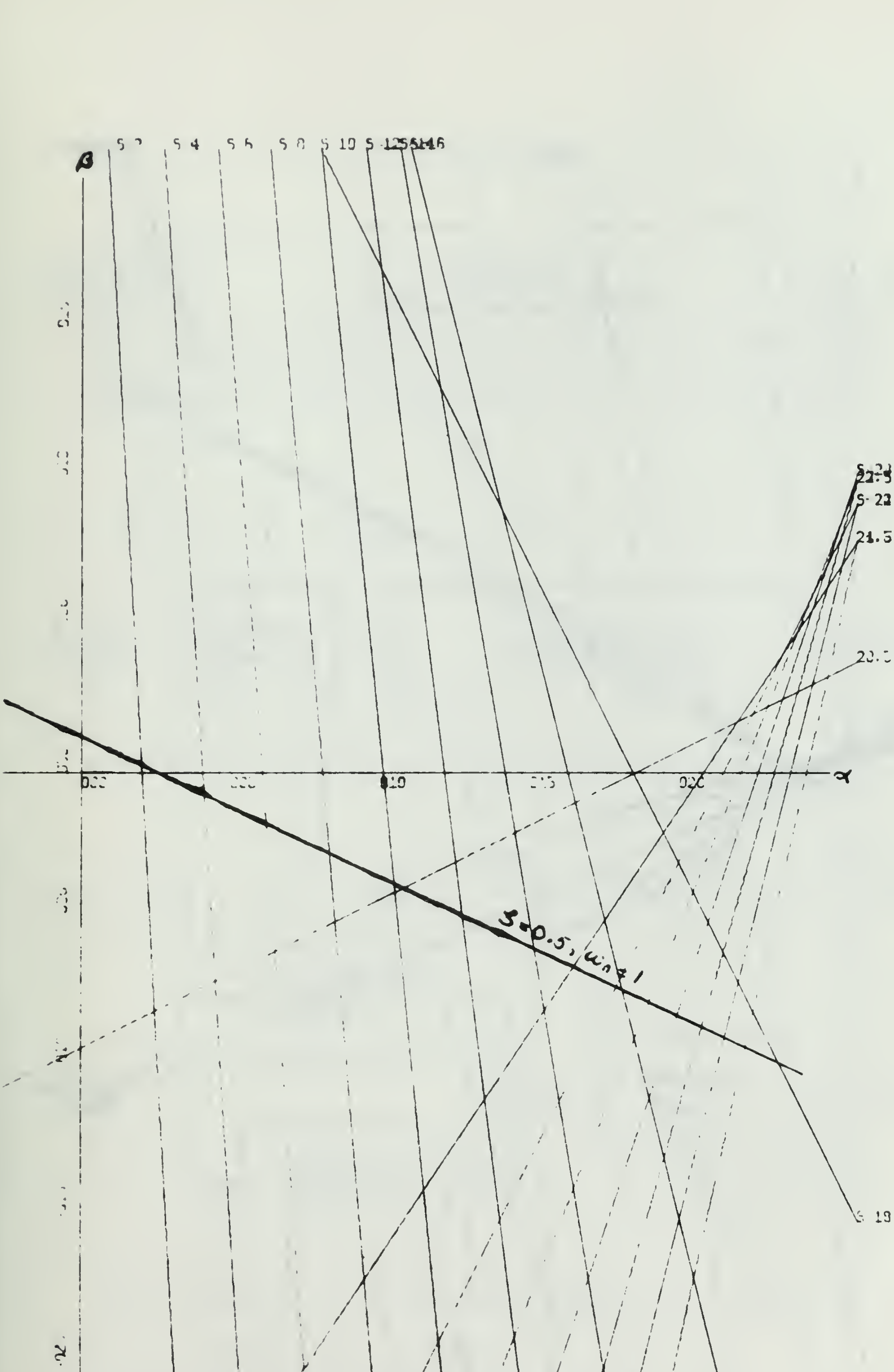
When a parameter appears in the numerator of an uncompensated system, the technique used in the previous examples must be modified to avoid an $\alpha\gamma$ or $\beta\gamma$ product. In chapter four it was shown that the plant numerator could be cancelled by the compensator, but in many situations this may not be desirable or even possible. However, there do exist systems, especially electrical or electronic, whose transfer function factors may be physically separable, so that the numerator parameter may be removed as a gain or filter component as shown in figures 5.13a and 5.13b. In this case the compensating function may then be introduced in the numerator of a parallel filter having the same denominator, as shown in figure 5.13c, where the compensated characteristic equation is

$$s^4 + (\alpha+2)s^3 + (2\alpha+100)s^2 + (100\alpha+\beta+\gamma)s + 2\beta + 0.1\gamma = 0 \qquad (5\text{-}64)$$

if k = 0.1, a completely random value. Then for $\zeta = 0.5$, $\omega_n = 1$,

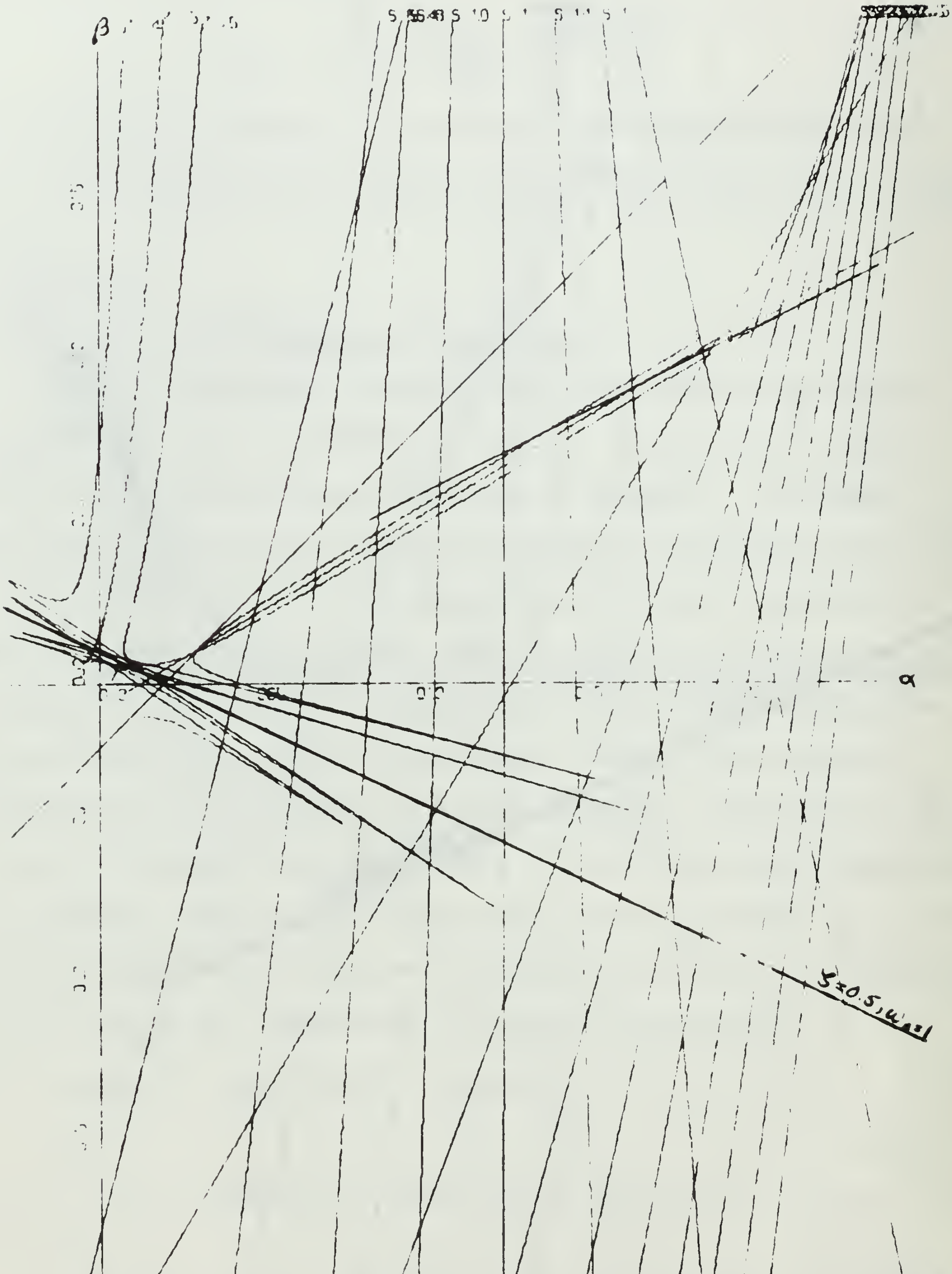
$$B_1 = 1 \quad C_1 = -2 \quad D_1 = -0.1 \quad F_1 = 98 \qquad (5\text{-}65)$$

$$B_2 = -98 \quad C_2 = -1 \quad D_2 = -1 \quad F_2 = 99. \qquad (5\text{-}66)$$



Scale:α=5/in.,β=5/in.

Fig. 5.11 Example X-Reduced Characteristic Equation



Scale:α=5/in.,β=5/in.

Fig. 5.12 Example X-Characteristic Equation

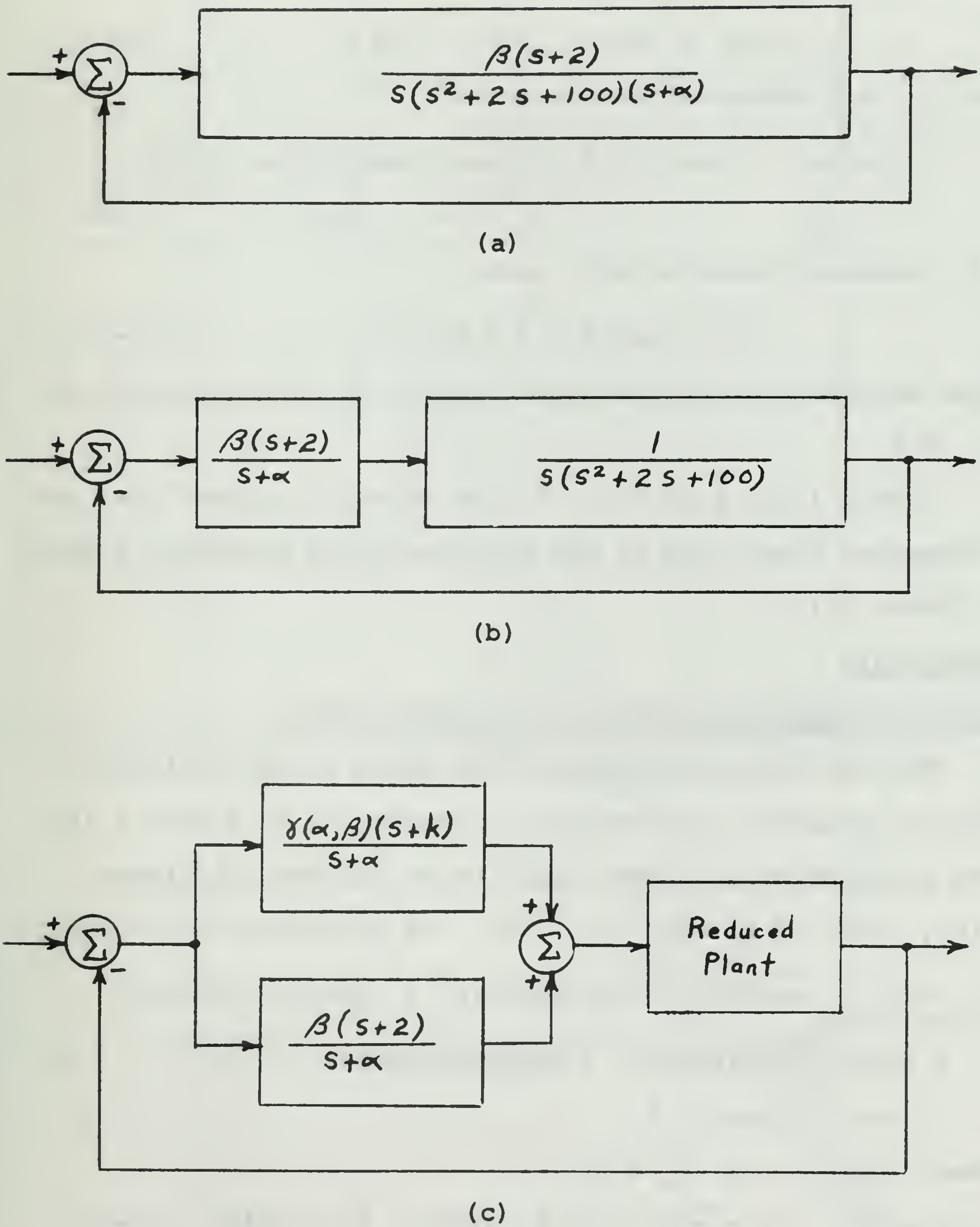


Fig. 5.13 Example XI-Type 1, 4th Order Plant

The singular line is

$$10.8\alpha - 1.9\beta + 88.1 = 0. \qquad (5\text{-}67)$$

The compensating function

$$\gamma(\alpha,\beta) = -44\alpha - 10.5\beta + 539.5 \qquad (5\text{-}68)$$

results in a characteristic equation

$$s^4 + (\alpha+2)s^3 + (2\alpha+100)s^2 + (56\alpha-9.5\beta+539.5)s - 4.4\alpha + 0.95\beta + 53.95 = 0 \qquad (5\text{-}69)$$

and a reduced characteristic equation

$$s^2 + (\alpha+1)s + \alpha + 98 = 0 \qquad (5\text{-}70)$$

which has left-half plane roots for $\alpha > -1$ and real roots for $\alpha > 20.8$.

A root locus plot for $\alpha > 0$ is shown in figure 5.14, and a parameter plane plot of the characteristic equation is shown in figure 5.15.

Example XII

Type 2-6th Order-Parameter in Plant Numerator

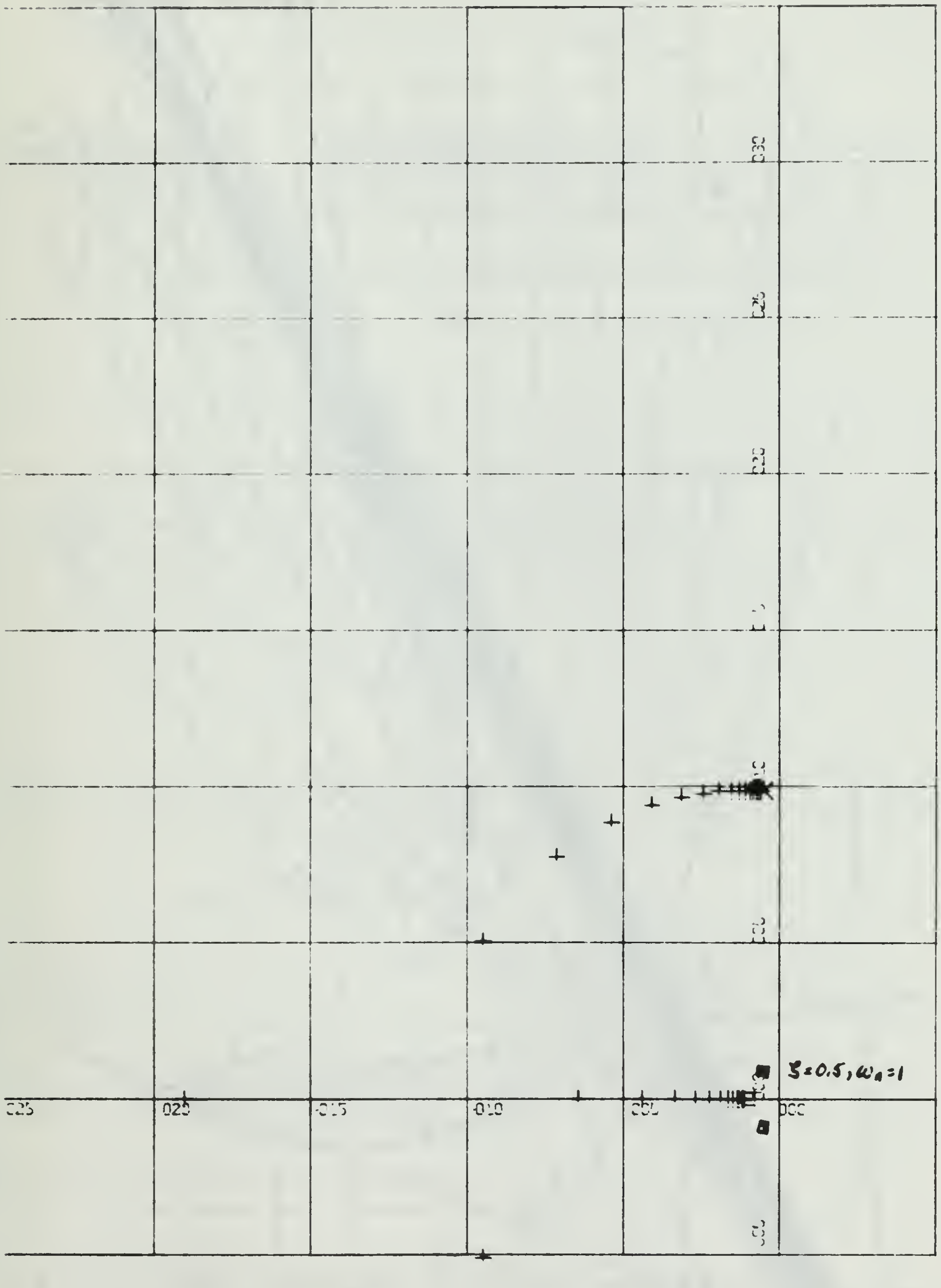
For the system of figure 5.16a again assume it is possible to separate the numerator parameter as in figure 5.16b. Then a compensation scheme could be as depicted in figure 5.16c, where if $k_1 = 10$, $k_2 = 100$, the characteristic equation

$$s^7 + 40s^6 + (\alpha+600)s^5 + (30\alpha+4000)s^4 + (300\alpha+\beta+\gamma+10000)s^3 + (1000\alpha+20\beta+1010\gamma)s^2 + (150\beta+10050\gamma)s + 500\beta + 50000\gamma = 0 \qquad (5\text{-}71)$$

gives, for $\zeta = 0.5$, $\omega_n = 1$,

$$B_1 = 701 \quad C_1 = -481 \quad D_1 = -48991 \quad F_1 = -9440 \qquad (5\text{-}72)$$

$$B_2 = 971 \quad C_2 = -130 \quad D_2 = -9040 \quad F_2 = -3401. \qquad (5\text{-}73)$$



Scale:X=5/in.,Y=5/in.

Fig. 5.14 Example XI-Reduced Characteristic Equation Root Locus

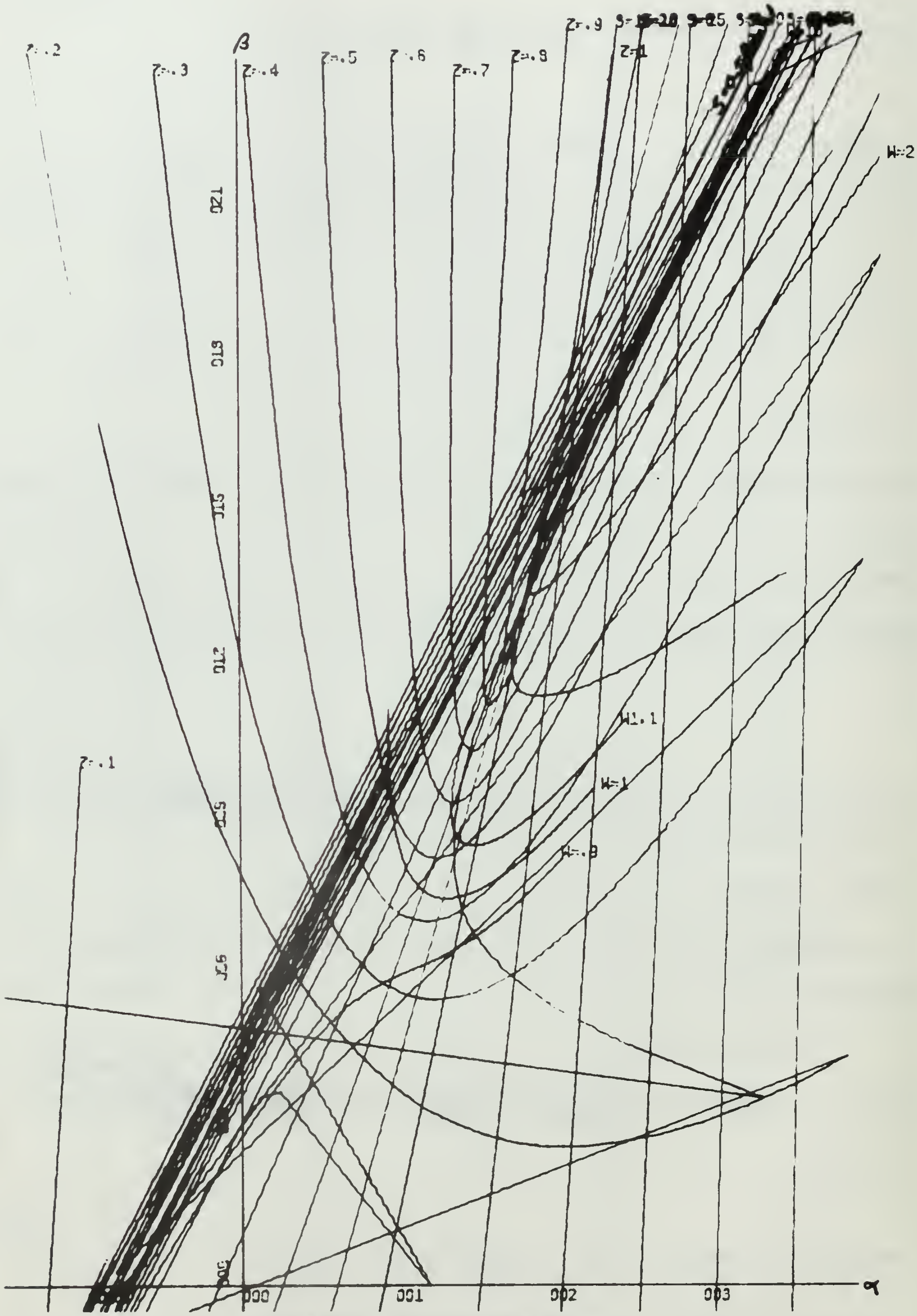
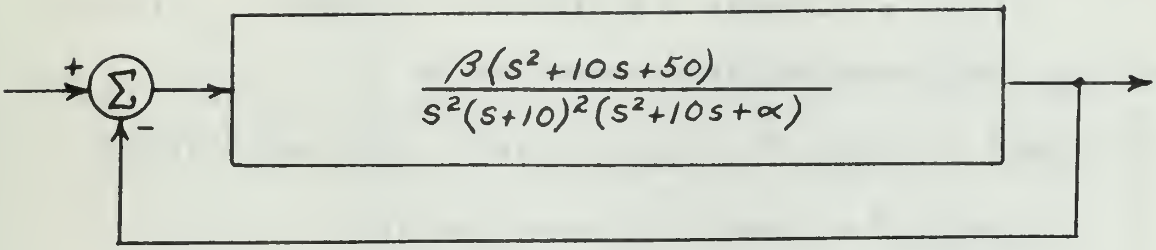
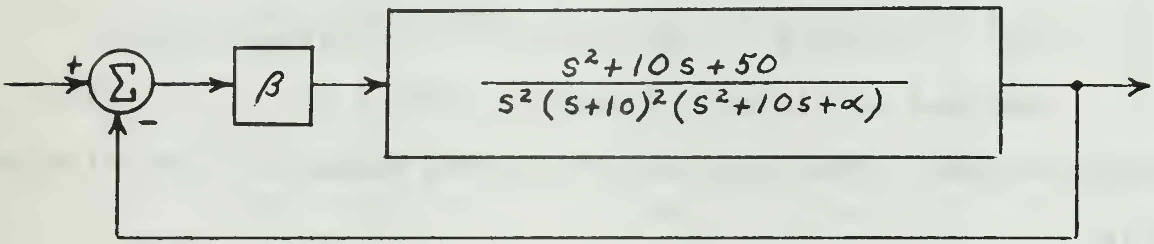


Fig. 5.15 Example XI-Characteristic Equation



(a)



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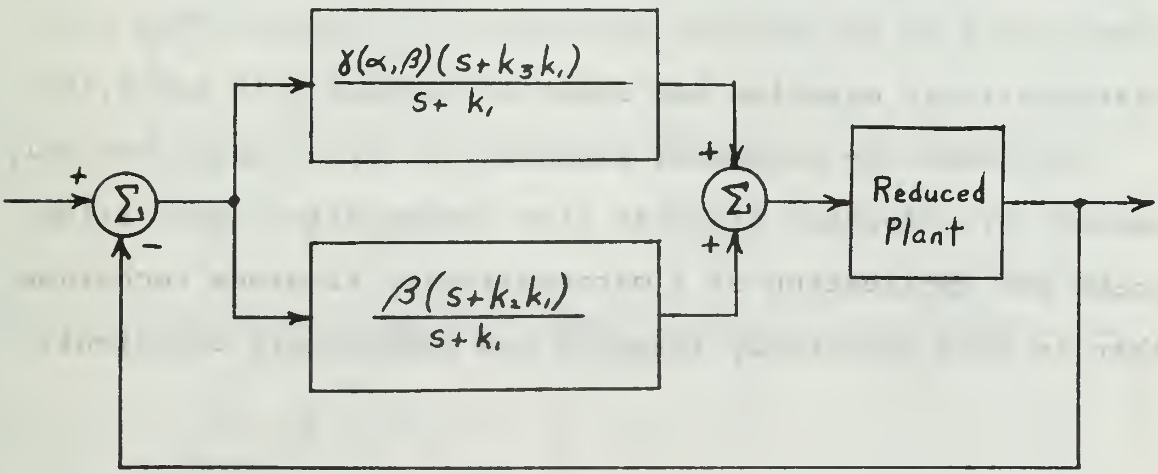


(c)

Fig. 5.16 Example XII-Type 2, 6th Order Plant

The singular line is

$$23.2\alpha - \beta - 40.2 = 0. \tag{5-74}$$

$$\gamma(\alpha,\beta) = 0.06091\alpha - 0.012055\beta - 0.28465. \tag{5-75}$$

The resultant characteristic equation is

$$\begin{aligned} s^7 &+ 40s^6 + (\alpha+600)s^5 + (30\alpha+4000)s^4 + (300.06\alpha+0.9879\beta \\ &+10000)s^3 + (1061.5\alpha+7.8244\beta-287.5)s^2 \\ &+ (612.15\alpha+28.847\beta-2861)s + 3045.5\alpha - 102.8\beta \\ &- 14230 = 0; \end{aligned} \tag{5-76}$$

and the reduced characteristic equation is

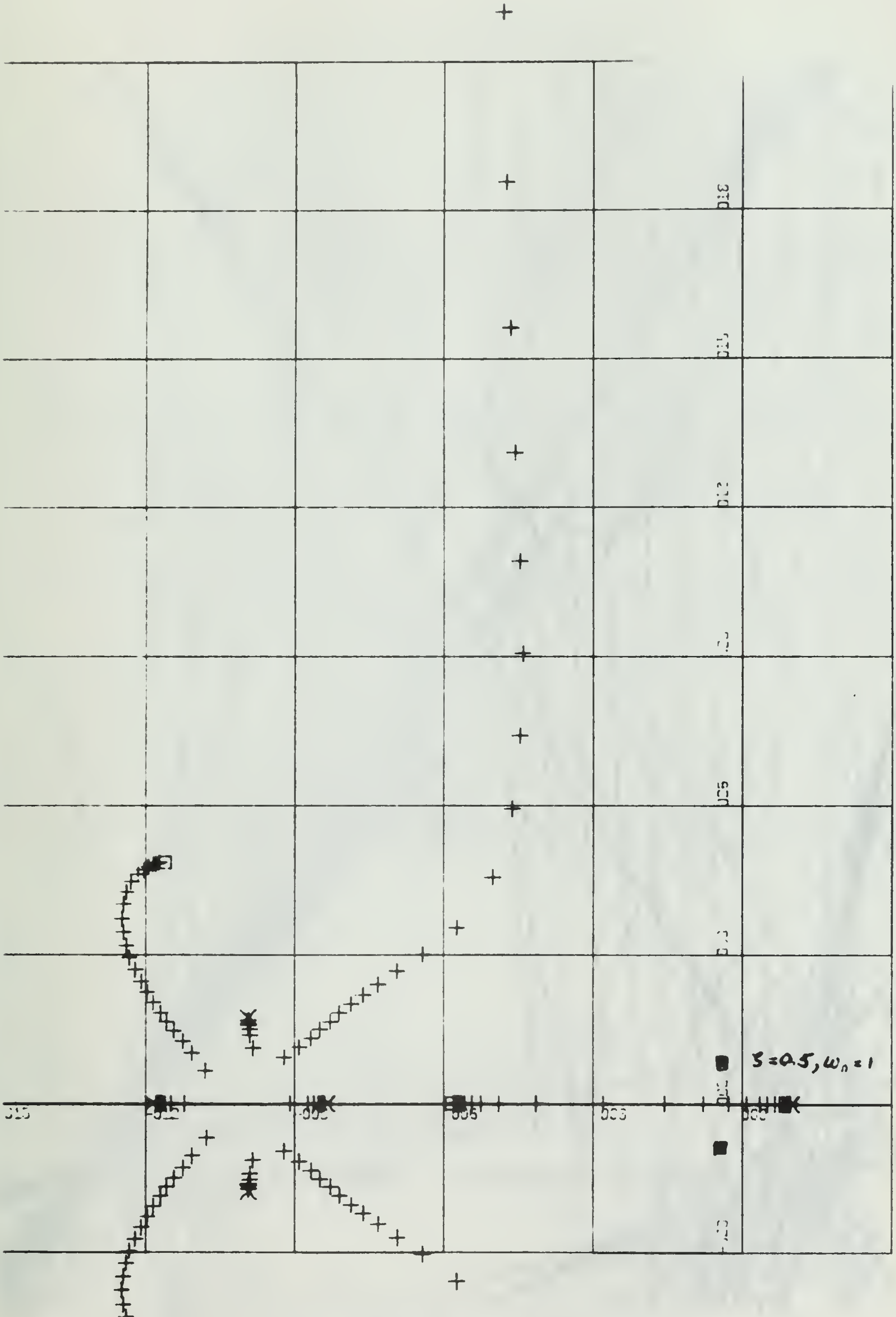
$$\begin{aligned} s^5 &+ 39s^4 + (\alpha+560)s^3 + (29\alpha+3401)s^2 + (270.06\alpha+0.9879\beta \\ &+6039)s + 761.44\alpha + 6.8365\beta - 9728 = 0, \end{aligned} \tag{5-77}$$

which becomes, after application of the singular line relationship,

$$\begin{aligned} s^5 &+ 39s^4 + (\alpha+560)s^3 + (29\alpha+3401)s^2 + (293.03\alpha+5999.3)s \\ &+ 920.73\alpha - 10000 = 0, \end{aligned} \tag{5-78}$$

for which a root locus plot is shown in figure 5.17. Parameter plane plots of the reduced characteristic equation and the characteristic equation are shown in figures 5.18 and 5.19.

Although the preceding examples are by no means the only method for effecting singular line compensation, they illustrate the application of a mathematically rigorous technique that is both physically feasible and laboriously expedient.

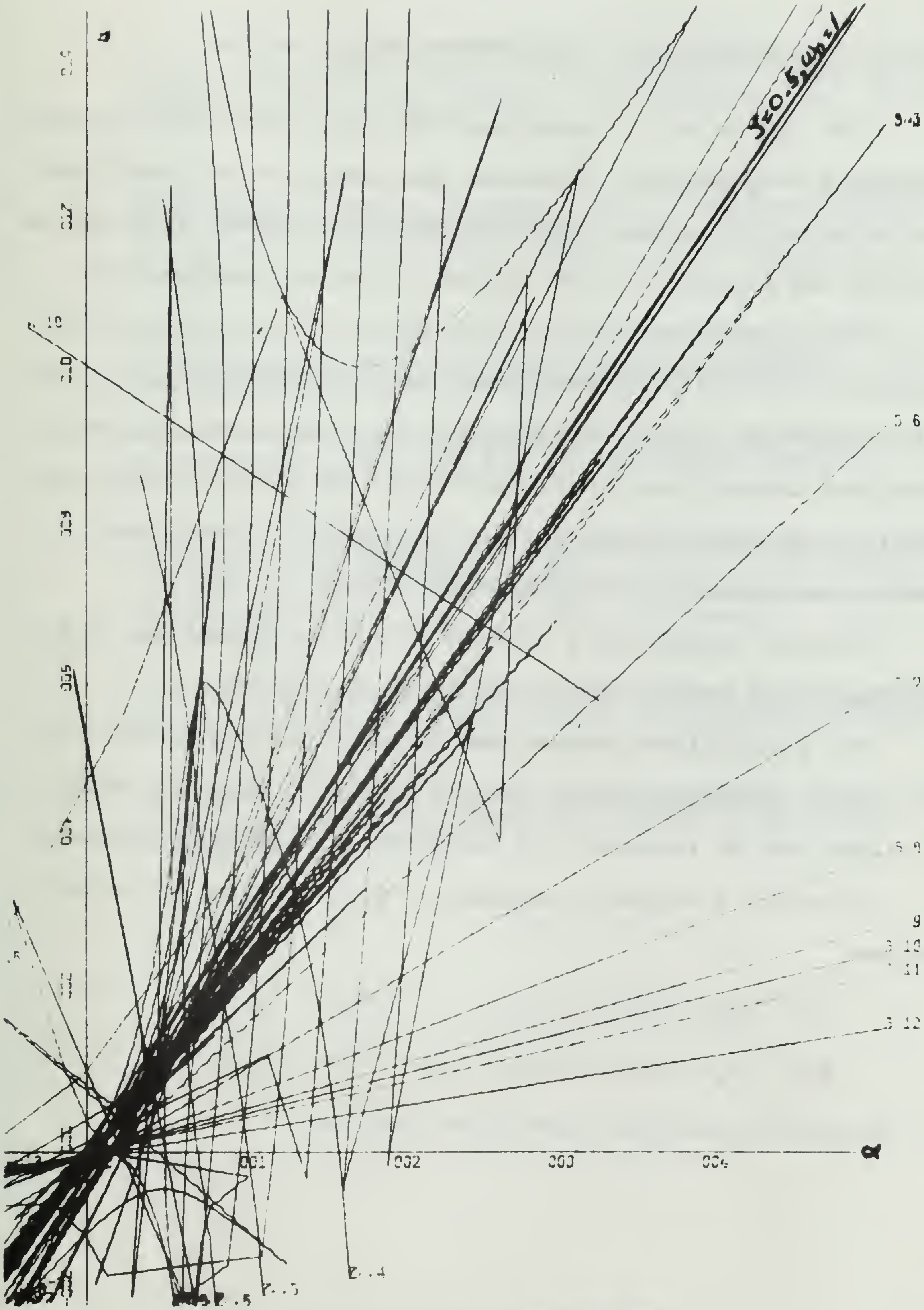


Scale:X=3/in.,Y=3/in.

Fig. 5.17 Example XII-Reduced Characteristic Equation Root Locus

Scale:α=100/in.,β=2000/in.

Fig. 5.18 Example X.I-Reduced Characteristic Equation



Scale:α=100/in.,β=2000/in.

Fig. 5.19 Example XII-Characteristic Equation

CHAPTER VI

RECOMMENDATIONS FOR FURTHER INVESTIGATION

This thesis in no manner exhausts the study of parameter space and singularity. Although the compensation techniques can be extended to many types of control systems, it would be foolish and presumptuous to assume universal applicability.

Solid state operational amplifiers offer an attractive means for synthesizing the compensators discussed in the previous chapters, but examination of the compensator transfer functions reveal that truly self-adaptive systems cannot be realized by these means until a voltage regulated linear resistance device can be achieved.

Digital compensating devices should be investigated for systems where expense is not an overriding factor.

The possibility exists that Cadena's method of folding the third co-ordinate into the α-β plane to obtain a grid of γ values may be repeated n-2 times for n parameter problems.

Finally, a computer program to analytically solve the system

$$B_1\alpha + C_1\beta + D_1\gamma + E_1\delta + \dots + N_1\nu + P_1 = 0 \qquad (6\text{-}1)$$

$$B_2\alpha + C_2\beta + D_2\gamma + E_2\delta + \dots + N_2\nu + P_2 = 0 \qquad (6\text{-}2)$$

by iterative processes should be possible.

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APPENDIX I

Table of Values of U_k (0.5)

k	U_k (0.5)
-1	-1
0	0
1	1
2	1
3	0
4	-1
5	-1
6	0
7	1
8	1
9	0
10	-1

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ABSTRACT

Previous work on parameter plane, three dimensional parameter space, and singular lines in the parameter plane are reviewed.

A general concept of n-dimensional parameter space is hypothesized whereby the parameter plane becomes a special case of the general hypothesis. By the same argument the singular line is shown to be a special case of the singular hyperplane.

Existence criteria for singular lines are established, and compensation methods for creating singular lines in non-singular systems are derived and used.

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